The running intersection relaxation of the multilinear polytope *

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Abstract

The multilinear polytope MP_G of a hypergraph $G = (V, E)$ is the convex hull of a set of binary points $z \in \{0,1\}^{V+E}$ satisfying a collection of multilinear equations of the form $z_e = \prod_{v \in e} z_v$ for all $e \in E$. We introduce the running intersection inequalities, a new class of facet-defining inequalities for the multilinear polytope. Accordingly, we define a new polyhedral relaxation of MP_G referred to as the running intersection relaxation and identify conditions under which this relaxation is sharp. Namely, we show that for β-acyclic hypergraphs with the simple intersection property, a class that lies between γ-acyclic and β-acyclic hypergraphs, the polytope MP_G admits a polynomial-size extended formulation whose projection onto the original space coincides with the running intersection relaxation.

Key words: multilinear polytope; running intersection property; hypergraph acyclicity; mixed-integer nonlinear optimization; polyhedral relaxations; extended formulations

1 Introduction

Multilinear sets and polytopes. Consider a hypergraph $G = (V, E)$, where V is the set of nodes of G, and E is a set of subsets of V of cardinality at least two, called the edges of G. With any hypergraph G, we associate a *multilinear set* $\mathcal{S}_G$ defined as:

$$\mathcal{S}_G = \Big\{ z \in \{0,1\}^{V+E} : z_e = \prod_{v \in e} z_v, \ \forall e \in E \Big\}.$$

Factorable reformulations of many types of mixed-integer nonlinear optimization problems (MINLP) contain a collection of multilinear equations of the form $z_e = \prod_{v \in e} z_v$. In particular, the set $\mathcal{S}_G$ represents the feasible region of linearized unconstrained $0-1$ polynomial optimization problems. Constructing strong polyhedral relaxations for multilinear sets has been a subject of extensive research by the mathematical optimization community [19, 8, 20, 21, 6, 16, 11, 10, 12, 9] and computational studies indicate that the quality of these relaxations has a significant impact on the performance of MINLP solvers [3, 2, 17, 18]. We refer to the convex hull of $\mathcal{S}_G$ as the *multilinear polytope* MP_G. If all multilinear equations defining $\mathcal{S}_G$ are bilinears, the multilinear polytope coincides with the Boolean quadric polytope defined by Padberg [19] in the context of unconstrained $0-1$ quadratic optimization. Indeed, the Boolean quadric polytope is a well-known polytope in combinatorial optimization and its facial structure has been thoroughly studied over the past three decades (see [13] for an exposition). In this paper, we consider multilinear sets containing higher degree multilinear equations.

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Literature review. The explicit characterization for the multilinear polytope of the complete hypergraph, namely, the hypergraph whose edge set consists of all subsets of nodes of cardinality at least two, has been presented in several independent studies (see for example [19, 22]). In [11] we study the facial structure of the multilinear polytope of general hypergraphs and develop the theory of a number of lifting operations, giving rise to many types of facet-defining inequalities. It is highly desirable, for both theoretical and numerical purposes, to obtain sufficient conditions under which a multilinear set is decomposable; namely, the multilinear polytope MP_G can be expressed as the intersection of a collection of multilinear polytopes MP_{G_j}, $j \in J$, where each hypergraph G_j has a simpler structure than the original hypergraph G. In [10] we provide necessary and sufficient conditions for decomposability of multilinear sets in terms of the structure of the intersection hypergraph. Padberg [19] shows that the Boolean quadric polytope admits a simple compact description when the corresponding graph is acyclic. The notion of graph acyclicity has been extended to several different notions of hypergraph acyclicity; in increasing order of generality, one can name Berge-acyclicity, γ-acyclicity, and β-acyclicity. In [12], we study the complexity of the facet-description of the multilinear polytope in conjunction with the "acyclicity degree" of its underlying hypergraph. In particular, we provide explicit characterizations of the multilinear polytopes of Berge-acyclic and γ-acyclic hypergraphs. Subsequently, as the multilinear polytope of γ-acyclic hypergraphs may contain exponentially many facets, we present a strongly polynomial-time algorithm to solve the separation problem. As a byproduct, we introduce "flower inequalities" which generalize 2-link inequalities defined in [9] corresponding to Berge-cycles of length two.

Our contribution. In this paper we introduce a new class of facet-defining inequalities for the multilinear polytope which can be utilized to construct stronger polyhedral relaxations for multilinear sets of general hypergraphs. The proposed inequalities, referred to as "running intersection inequalities," serve as a significant generalization of flower inequalities [12] for the multilinear polytope of γ-acyclic hypergraphs. Subsequently, we define the running intersection relaxation, a new polyhedral relaxation for the multilinear set obtained by adding all running intersection inequalities to its standard linearization. We show that for β-acyclic hypergraphs with the "simple intersection property", a class that lies between γ-acyclic hypergraphs and β-acyclic hypergraphs, the multilinear polytope admits a compact extended formulation whose projection onto the original space coincides with the running intersection relaxation. More precisely, for a β-acyclic hypergraph $G = (V, E)$ with the simple intersection property and with edges of cardinality at most r, the proposed extended formulation has at most $|V| + 2|E|$ variables and $2(|V| + (r+2)|E|)$ inequalities, while the multilinear polytope in the original space may contain exponentially many facet-defining inequalities. This in turn implies that optimizing a linear function over MP_G can be done in polynomial time. The proposed extended formulation is obtained by showing that, after the addition of at most $|E|$ edges to the original hypergraph G, the corresponding multilinear set is decomposable into a collection of multilinear sets whose convex hulls have compact descriptions. To this end, we present a new sufficient condition for decomposability of multilinear sets, a result which is of independent interest.

Organization. In Section 2 we introduce running intersection inequalities, we establish some of their basic properties, and we identify conditions under which they induce facets of the multilinear polytope. In Section 3 we show that the running intersection relaxation coincides with the multilinear polytope of β-acyclic hypergraphs with the simple intersection property. Finally, proofs of the main theorems are given in Section 4.

2 The running intersection relaxation

We start by formally defining a hypergraph. A *hypergraph* G is a pair (V, E), where V is a finite set of nodes and E is a multiset of subsets of V, called the edges of G. Unless stated otherwise, throughout this paper we consider hypergraphs without loops or parallel edges, in which case E is a set of subsets of V of cardinality at least two. We refer to the node set of G as $V(G)$ and to the edge set of G as $E(G)$. We define the *support hypergraph* of a valid inequality $az \leq \alpha$ for MP_G, as the hypergraph $G(a)$, where $V(G(a)) = \{v \in V : a_v \neq 0\} \cup (\cup_{e \in E: a_e \neq 0} e)$, and $E(G(a)) = \{e \in E : a_e \neq 0\}$. In [12] we introduced flower inequalities, a class of facet-defining inequalities for the multilinear polytope whose support hypergraphs are γ-acyclic. In this section, we present a significant generalization of these inequalities that does not require γ-acyclicity of the support hypergraph. To obtain the new cutting planes, we make use of the notion of running intersection property, which was introduced in the database community to study acyclic databases [4] and has been used extensively by the machine learning community to infer conditional independence in graphical models [15].

2.1 The running intersection property

A multiset F of subsets of a finite set V has the *running intersection property* if there exists an ordering $p_1, p_2, \ldots, p_m$ of the sets in F such that

$$\text{for each } k = 2, \ldots, m, \text{ there exists } j < k \text{ such that } p_k \cap \Big(\bigcup_{i<k} p_i\Big) \subseteq p_j. \tag{1}$$

Throughout the paper, we refer to an ordering $p_1, p_2, \ldots, p_m$ satisfying (1) as a *running intersection ordering* of F. Each running intersection ordering $p_1, p_2, \ldots, p_m$ of F induces a collection of sets

$$N(p_1) := \emptyset, \qquad N(p_k) := p_k \cap \Big(\bigcup_{i<k} p_i\Big) \text{ for } k = 2, \ldots, m. \tag{2}$$

It can be shown that if a multiset F with $|F| \geq 2$ has the running intersection property, then there exist several running intersection orderings of F. We refer to an element $f \in F$ as a *leaf* of F if there exists a running intersection ordering of F in which f is the last element. The following lemma states some basic properties of multisets with the running intersection property and has been stated in various forms in the literature (see for example [4]).

Lemma 1. *Let F be a multiset with the running intersection property. If $|F| \geq 2$, then*

(i) F has at least two leaves,

(ii) for any $f \in F$, there exists a running intersection ordering of F in which f is the first element,

(iii) for any $f \in F$ such that $f \subseteq f'$ for some $f' \in F$, the multiset $F \setminus \{f\}$ has the running intersection property.

As we detail in the following, to obtain running intersection inequalities we make use of the number of connected components of a related hypergraph. We now formalize the concept of hypergraph connectivity. We first present the notion of a chain in a hypergraph as defined in [5]. A *chain* in G of length t for some $t \geq 1$, is a sequence $P = v_1, e_1, v_2, e_2, \ldots, e_t, v_{t+1}$ such that $v_1, v_2, \ldots, v_t$ are distinct nodes of G, $e_1, e_2, \ldots, e_t$ are distinct edges of G, and $v_i, v_{i+1} \in e_i$ for $i = 1, \ldots, t$. A hypergraph G is *connected* if for any two distinct nodes v_i, v_j of G, there is a chain between v_i and v_j in G. Consider a hypergraph $G = (V, E)$ and let V' be a subset of V. A hypergraph (V', E') is a *partial hypergraph* of G if $E' \subseteq E$. The *section hypergraph* of G induced by V' is the partial hypergraph (V', E'), where $E' = \{e \in E : e \subseteq V'\}$. The *connected components* of G are the maximal connected section hypergraphs of G. We refer to a node of G as an *isolated node* if it is not contained in any edge of G. Note that an isolated node corresponds to a connected component. The next lemma provides an alternative characterization for the number of connected components of a hypergraph whose edge set has the running intersection property.

Lemma 2. *Let $G = (V, E)$ be a hypergraph. Assume that there exists a running intersection ordering $e_1, \ldots, e_m$ of E and denote by $N(e_1), \ldots, N(e_m)$ the corresponding sets defined in (2). Then the number of connected components of G is*

$$n_0 + |\{e \in E : N(e) = \emptyset\}|,$$

where n_0 is the number of isolated nodes of G.

Proof. To prove the statement, it suffices to show that the number ω of connected components of a hypergraph G with no isolated nodes is $|\{e \in E : N(e) = \emptyset\}|$. The proof is by induction on $m = |E|$, the base case being trivial. Let $G' = (V', E')$ be the hypergraph with node set $V' := \cup_{k=1}^{m-1} e_k$ and edge set $E' := \{e_1, \ldots, e_{m-1}\}$. Note that $e_1, \ldots, e_{m-1}$ is a running intersection ordering of E' and that the corresponding sets (2) are $N'(e_k) = N(e_k)$ for all $k = 1, \ldots, m-1$. Thus by induction the number ω' of connected components of G' is $|\{e \in E' : N(e) = \emptyset\}|$. First consider the case $e_m \cap E' = \emptyset$. In this case $N(e_m) = \emptyset$ and G has one more connected component than G'; that is,

$$\omega = \omega' + 1 = |\{e \in E' : N(e) = \emptyset\}| + 1 = |\{e \in E : N(e) = \emptyset\}|.$$

Next, consider the case $e_m \cap E' \neq \emptyset$. It then follows that $N(e_m) \neq \emptyset$ and G has the same number of connected component as G'. Thus

$$\omega = \omega' = |\{e \in E' : N(e) = \emptyset\}| = |\{e \in E : N(e) = \emptyset\}|.$$

□

We are now in position to define running intersection inequalities.

2.2 Running intersection inequalities

Consider a hypergraph $G = (V, E)$. Let $e_0 \in E$ and let e_k, $k \in K$, be a collection of edges adjacent to e_0 in G such that the multiset

$$\tilde{E} := \{e_0 \cap e_k : k \in K\} \tag{3}$$

has the running intersection property. Consider a running intersection ordering of $\tilde{E}$ with the corresponding sets $N(e_0 \cap e_k)$, for $k \in K$, as defined in (2). For each $k \in K$ with $N(e_0 \cap e_k) \neq \emptyset$, let u_k be a node in $N(e_0 \cap e_k)$. We define a *running intersection inequality* as

$$-\sum_{k \in K : N(e_0 \cap e_k) \neq \emptyset} z_{u_k} + \sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} z_v + \sum_{k \in K} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{4}$$

where ω is the number of connected components of the hypergraph $\tilde{G} = (e_0, \tilde{E})$. We refer to e_0 as the *center* and to e_k, $k \in K$, as the *neighbors*. Note that unlike G, the hypergraph $\tilde{G}$ may have loops and parallel edges. By Lemma 2, the right-hand side of (4) is equal to the sum of the coefficients of the left-hand side. In the special case where $N(e_0 \cap e_k) = \emptyset$ for all $k \in K$ running intersection inequalities simplify to flower inequalities introduced in [12].

We now establish the validity of running intersection inequalities for MP_G.

Proposition 1. *Running intersection inequalities are valid for the multilinear polytope.*

Proof. Consider a running intersection inequality (4). Let $\tilde{G} = (e_0, \tilde{E})$ be the corresponding hypergraph where $\tilde{E}$ is defined by (3), and let $\mathcal{O}$ denote a running intersection ordering of $\tilde{E}$ with the sets $N(e_0 \cap e_k)$, $k \in K$, as defined in (2).

Denote by $\tilde{G}_i$, for $i = 1, \ldots, \omega$, the connected components of $\tilde{G}$. For each $\tilde{G}_i$, define $K_i = \{k \in K : e_k \cap e_0 \in E(\tilde{G}_i)\}$. Clearly, the sets K_i, for $i = 1, \ldots, \omega$, form a partition of K. We argue that for each $\tilde{G}_i$ with $K_i \neq \emptyset$, the following inequality is valid for MP_G.

$$-\sum_{k \in K_i : N(e_0 \cap e_k) \neq \emptyset} z_{u_k} + \sum_{k \in K_i} z_{e_k} \leq 1. \tag{5}$$

If $|K_i| = 1$, say $K_i = \{1\}$, then $N(e_0 \cap e_1) = \emptyset$; thus the validity of (5) is trivial. Henceforth, assume that $|K_i| \geq 2$. We claim that the maximum value of the left-hand side of inequality (5) is one, and this value is attained if and only if $z_{e_k} = 1$ for all $k \in K_i$. Let $\mathcal{O}_i$ be the subsequence of $\mathcal{O}$ corresponding to the edges $e_0 \cap e_k$, with $k \in K_i$. It can be shown that $\mathcal{O}_i$ is a running intersection ordering of $E(\tilde{G}_i)$. Without loss of generality, let $\mathcal{O}_i = e_0 \cap e_1, e_0 \cap e_2, \ldots, e_0 \cap e_t$, where $t := |E(\tilde{G}_i)|$. Since $\tilde{G}_i$ is a connected hypergraph by Lemma 2, we have $N(e_0 \cap e_k) \neq \emptyset$ for all $k = 2, \ldots, t$. This implies that for each $k = 2, \ldots, t$, the node u_k exists and if $z_{e_k} = 1$, we have $z_{u_k} = 1$. Consequently the value of the left-hand side of inequality (5) is at most one and if it is equal to one, we must have $z_{e_1} = 1$. Now suppose that $z_{e_1} = 1$. Since $u_2 \in e_1$, it follows that $z_{u_2} = 1$. Hence, if the maximum value of the left-hand side of (5) is attained, we must have $z_{e_2} = 1$. If $t = 2$, the proof is complete. Otherwise, since u_3 is in e_1 or in e_2 and $z_{e_1} = z_{e_2} = 1$, we have $z_{u_3} = 1$ which in turn implies $z_{e_3} = 1$. Hence, by a recursive application of this argument for each element of $\mathcal{O}_i$, we conclude that inequality (5) is binding if and only if $z_{e_k} = 1$ for all $k \in K_i$.

By summing up inequalities (5) for all $\tilde{G}_i$ with $E(\tilde{G}_i) \neq \emptyset$ together with inequalities $z_{v_i} \leq 1$ for all $\tilde{G}_i$ with $V(\tilde{G}_i) = \{v_i\}$ and $E(\tilde{G}_i) = \emptyset$, we conclude that the value of the three summations on the left hand side of (4) does not exceed ω. In addition, this maximum value is attained only if $z_{e_k} = 1$ for all $k \in K$ and $z_v = 1$ for all $v \in e_0 \setminus (\cup_{k \in K} e_k)$ which in turn implies $z_{e_0} = 1$. Hence, inequality (4) is valid. □

Example 1. *Consider the hypergraph $G = (V, E)$ with $V = \{v_1, \ldots, v_7\}$ and $E = \{e_0, e_1, e_2, e_3, e_4\}$, where we define $e_0 := V$, $e_1 := \{v_1, v_2, v_3, v_7\}$, $e_2 := \{v_2, v_3, v_6\}$, $e_3 := \{v_1, v_3, v_5\}$, $e_4 := \{v_1, v_2, v_4\}$. Consider the set $\tilde{E} = \{e \cap e_0 : e \in E \setminus e_0\}$. It is then simple to see that the sequence $\mathcal{O} = e_1, e_2, e_3, e_4$ is a running intersection ordering of $\tilde{E}$. By (2) we have $N(e_0 \cap e_4) = \{v_1, v_2\}$, $N(e_0 \cap e_3) = \{v_1, v_3\}$, $N(e_0 \cap e_2) = \{v_2, v_3\}$. Hence, the system of running intersection inequalities centered at e_0 with neighbors $E \setminus \{e_0\}$ is given by*

$$-2z_{v_i} - z_{v_j} - z_{e_0} + z_{e_1} + z_{e_2} + z_{e_3} + z_{e_4} \leq 0 \qquad \text{for all distinct pairs } (i, j) \in \{1, 2, 3\}. \tag{6}$$

It can be checked that all of the above inequalities define facets of MP_G. Note that one can write many more running intersection inequalities for MP_G. Due to space limitations, we only listed those centered at e_0 with neighbors $E \setminus \{e_0\}$. ◊

Consider the set of all running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$. To construct these inequalities, we make use of a running intersection ordering of the multiset $\tilde{E}$ defined by (3), and by Lemma 1, such an ordering is not unique. However, the following proposition implies that the system of all running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$, is independent of the running intersection ordering.

Proposition 2. *Let F be a multiset with the running intersection property. Then any running intersection ordering of F leads to the same multiset $\{N(e) : e \in F\}$ as defined in* (2)*.*

Proof. We prove the statement by induction on $|F|$. Given a multiset F' of subsets of a finite set and $e, f \in F'$, we say that e is a *parent* of f in F' if $f \cap (\bigcup_{g \in F' \setminus f} g) \subseteq e$.

In the base case we have $|F| = 1$; the running intersection ordering is unique and thus the statement trivially follows. We also consider the base case $|F| = 2$. Let $f, g \in F$. If $f \cap g = \emptyset$, then independent of the running intersection ordering, we obtain $N(f) = N(g) = \emptyset$. Thus, we assume that $f \cap g$ is nonempty. Let $\mathcal{O}$ be a running intersection ordering of F. If $\mathcal{O} = g, f$, we obtain $N(f) = f \cap g$ and $N(g) = \emptyset$. Viceversa, if $\mathcal{O} = f, g$, we obtain $N'(g) = g \cap f$ and $N'(f) = \emptyset$. Hence the two multisets coincide. Note that in the latter base case, even though the two multisets coincide, the function that associates to each $e \in F$ the set $N(e)$ is not independent of the running intersection ordering.

We now prove the inductive step. Let $\mathcal{O}$ and $\mathcal{O}'$ be two running intersection orderings of F. Let $\{N(e) : e \in F\}$ be the multiset corresponding to $\mathcal{O}$ and let $\{N'(e) : e \in F\}$ be the multiset corresponding to $\mathcal{O}'$. If the last set in $\mathcal{O}$ and $\mathcal{O}'$ is the same set, say f, then we have $N(f) = N'(f)$. By dropping the last set from $\mathcal{O}$ and $\mathcal{O}'$ we obtain two running intersection orderings $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}'$ of $F \setminus \{f\}$, respectively. By induction the two multisets $\{N(e) : e \in F \setminus \{f\}\}$ and $\{N'(e) : e \in F \setminus \{f\}\}$ coincide, hence also the multisets $\{N(e) : e \in F\}$ and $\{N'(e) : e \in F\}$ coincide. Thus we now assume that the last set in $\mathcal{O}$, say f, is different from the last set in $\mathcal{O}'$, say g.

Since f and g are leaves of F, they both have a parent in F. Let $p(f)$ be a parent of f in F, and let $p(g)$ be a parent of g in F. Note that there might be several sets of F that are parents of f. If g is a parent of f, then we set $p(f) := g$. Symmetrically, if f is a parent of g, then we set $p(g) := f$.

We first consider the case where $p(f) = g$ and $p(g) = f$. Since $p(g) = f$, for every set $e \in F \setminus \{f\}$ we have $g \cap e = f \cap g \cap e$. Let $\bar{F}$ be obtained from $F \setminus \{f\}$ by replacing the set g with a new set $f \cap g$ and let $\bar{\mathcal{O}}$ be obtained from $\mathcal{O}$ by dropping the last set f and by replacing g with $f \cap g$. Since by dropping the last set from $\mathcal{O}$ we obtain a running intersection ordering of $F \setminus \{f\}$, it can be checked that $\bar{\mathcal{O}}$ is a running intersection ordering of $\bar{F}$ and that the two running intersection orderings lead to the same multiset $\{N(e) : e \in F \setminus \{f\}\}$. Symmetrically, since $p(f) = g$, we define the set $\bar{F}'$ obtained from $F \setminus \{g\}$ by replacing the set f with a new set $f \cap g$. We also obtain $\bar{\mathcal{O}}'$ from $\mathcal{O}'$ by dropping the last set g and by replacing f with $f \cap g$. As above, $\bar{\mathcal{O}}'$ is a running intersection ordering of $\bar{F}'$. Moreover $\bar{\mathcal{O}}'$ and the running intersection ordering of $F \setminus \{g\}$ obtained by dropping the last set from $\mathcal{O}'$ lead to the same multiset $\{N'(e) : e \in F \setminus \{g\}\}$. Note that $\bar{F} = \bar{F}'$, thus by induction the two multiset $\{N(e) : e \in F \setminus \{f\}\}$ and $\{N'(e) : e \in F \setminus \{g\}\}$ coincide. Since $N(f) = f \cap g = N'(g)$, also the multisets $\{N(e) : e \in F\}$ and $\{N'(e) : e \in F\}$ coincide. This concludes the proof in the case $p(f) = g$ and $p(g) = f$.

We now assume that the assumption $p(g) = f$ and $p(f) = g$ does not hold. To study the multiset $\{N(e) : e \in F\}$ corresponding to $\mathcal{O}$, we define the multiset F_1 obtained from F by deleting the set f.

Claim 1. *If $p(g) \neq f$, then $p(g)$ is a parent of g in F_1. If $p(g) = f$, then $p(f)$ is a parent of g in F_1.*

Proof of claim. If $p(g) \neq f$, then $p(g)$ is a parent of g in F_1 since

$$g \cap (\cup_{e \in F \setminus \{f,g\}} e) \subseteq g \cap (\cup_{e \in F \setminus \{g\}} e) \subseteq p(g).$$

Assume now that $p(g) = f$. We have $g \cap (\cup_{e \in F \setminus \{f,g\}} e) \subseteq g \cap (\cup_{e \in F \setminus \{g\}} e) \subseteq f$, and $\cup_{e \in F \setminus \{f,g\}} e \subseteq \cup_{e \in F \setminus \{f\}} e$. Thus,

$$g \cap (\cup_{e \in F \setminus \{f,g\}} e) \subseteq f \cap (\cup_{e \in F \setminus \{f\}} e) \subseteq p(f).$$

Since $p(f) \neq g$, it follows that $p(f)$ is a parent of g in F_1. $\diamond$

By Claim 1, g has a parent in F_1. This implies that there exists a running intersection ordering of $F \setminus \{f\}$ with g as the last set. In fact, such a running intersection ordering can be obtained by appending g to a running intersection ordering of $F \setminus \{f, g\}$. Since by induction all running intersection orderings of $F \setminus \{f\}$ lead to the same multiset, we assume without loss of generality that the second to last set in $\mathcal{O}$ is g. We now explicitly write the obtained sets $N(f)$ and $N(g)$. To do so, we consider three cases: (A) $p(f) \neq g$ and $p(g) \neq f$, (B) $p(f) = g$ and $p(g) \neq f$, (C) $p(f) \neq g$ and $p(g) = f$.

Case (A). We have $N(f) = f \cap p(f)$ and by Claim 1 $N(g) = g \cap p(g)$.

Case (B). We have $N(f) = f \cap g$ and by Claim 1 $N(g) = g \cap p(g)$.

Case (C). We have $N(f) = f \cap p(f)$ and by Claim 1 $N(g) = g \cap p(f)$.

We now study the multiset $\{N'(e) : e \in F\}$ corresponding to $\mathcal{O}'$. Let F'_1 be obtained from F by deleting the set g. By Claim 1, with f and g permuted, and with F'_1 instead of F_1, we obtain

Claim 2. *If $p(f) \neq g$, then $p(f)$ is a parent of f in F'_1. If $p(f) = g$ then $p(g)$ is a parent of f in F'_1.*

By Claim 2, f has a parent in F'_1. This implies that there exists a running intersection ordering of $F \setminus \{g\}$ with f as the last set. Since by induction all running intersection orderings of $F \setminus \{g\}$ lead to the same multiset, we assume without loss of generality that the second to last set in $\mathcal{O}'$ is f. In order to explicitly write the obtained sets $N'(g)$ and $N'(f)$, we consider the three cases (A), (B), (C) introduced above.

Case (A). We have $N'(g) = g \cap p(g)$ and by Claim 2 $N'(f) = f \cap p(f)$.

Case (B). We have $N'(g) = g \cap p(g)$ and by Claim 2 $N'(f) = f \cap p(g)$.

Case (C). We have $N'(g) = g \cap f$ and by Claim 2 $N'(f) = f \cap p(f)$.

We now show that the multiset $\{N(f), N(g)\}$ equals the multiset $\{N'(g), N'(f)\}$. This concludes the proof of the proposition since the two orders obtained from $\mathcal{O}$ and $\mathcal{O}'$ by dropping the last two sets are running intersection orderings of the same set $F \setminus \{f, g\}$ and by induction the two corresponding multisets coincide.

Again, we consider the three cases (A), (B), (C). As the case (C) is symmetric to case (B), we will not consider it any further.

Case (A). We have $N(f) = N'(f)$, and $N(g) = N'(g)$. Thus we are done.

Case (B). We have $N(g) = N'(g)$. Thus we need to show $N(f) = f \cap g = f \cap p(g) = N'(f)$. Since $p(g)$ is a parent of g in F, we have $f \cap g \subseteq p(g)$, thus $f \cap g \subseteq f \cap p(g)$. Viceversa, since g is a parent of f in F, we have $f \cap p(g) \subseteq g$, thus $f \cap p(g) \subseteq f \cap g$. $\square$

By applying Proposition 2 to the multiset $\tilde{E}$ defined by (3), we obtain the following result.

Corollary 1. *Any running intersection ordering of $\tilde{E}$ leads to the same system of running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$.*

We now introduce a new polyhedral relaxation of multilinear sets. To this end, we first recall a widely-used polyhedral relaxation of $\mathcal{S}_G$ which is obtained by replacing each multilinear equation

$z_e = \prod_{v \in e} z_v$, by its convex hull over the unit hypercube:

$$\begin{aligned}
\mathrm{MP}_G^{\mathrm{LP}} := \Big\{ z : \quad & z_v \leq 1, \ \forall v \in V, \\
& z_e \geq 0, \ z_e \geq \sum_{v \in e} z_v - |e| + 1, \ \forall e \in E, \\
& z_e \leq z_v, \ \forall e \in E, \ \forall v \in e \Big\}.
\end{aligned} \tag{7}$$

The above relaxation has been used extensively in the literature and is often referred to as the *standard linearization* of the multilinear set (see, e.g., [8]). We define the *running intersection relaxation* of $\mathcal{S}_G$, denoted by $\mathrm{MP}_G^{\mathrm{RI}}$, as the polytope obtained by adding to $\mathrm{MP}_G^{\mathrm{LP}}$ all possible running intersection inequalities for $\mathcal{S}_G$. Note that running intersection inequalities with $K = \emptyset$ are already present in (7).

2.3 Redundant inequalities

For a general hypergraph G, many of the running intersection inequalities defined by (4) are redundant. The following proposition provides sufficient conditions to identify such redundant inequalities.

Proposition 3. *Every running intersection inequality centered at e_0 with neighbors e_k, $k \in K$, that defines a facet of MP_G^{RI} satisfies*

(i) $e_0 \cap e_k \not\subseteq e_0 \cap e_{k'}$ *for any* $k, k' \in K$,

(ii) $|e_0 \cap e_k| \geq 2$ *for all* $k \in K$, *and*

(iii) for any distinct $k, k' \in K$ *with* $u_k, u_{k'} \in N(e_0 \cap e_k) \cap N(e_0 \cap e_{k'})$, *we have* $u_k = u_{k'}$.

Proof. To prove the statement, we consider a running intersection inequality not satisfying each condition. Then we show that such an inequality can be obtained by summing up a number of other inequalities valid for $\mathrm{MP}_G^{\mathrm{RI}}$. Since MP_G is full dimensional [11], this implies that the inequality under consideration is not facet-defining.

Consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$. Assume that this inequality does not satisfy condition (i), i.e. there exist $i, j \in K$ such that $e_0 \cap e_i \subseteq e_0 \cap e_j$. Consider the multiset $\tilde{E}$ defined by (3). We show that there exists a running intersection ordering $\mathcal{O}$ of $\tilde{E}$ in which $e_0 \cap e_j$ appears before $e_0 \cap e_i$. Define $\tilde{E}' = \{e_0 \cap e_k : k \in K \setminus \{i\}\}$. Observe that by part (iii) of Lemma 1, the set $\tilde{E}'$ has the running intersection property. Consider a running intersection ordering $\mathcal{O}'$ of $\tilde{E}'$ and construct a sequence $\mathcal{O}$ obtained by inserting $e_0 \cap e_i$ right after $e_0 \cap e_j$ in $\mathcal{O}'$. It is now simple to check that $\mathcal{O}$ is a running intersection ordering of $\tilde{E}$. A running intersection inequality centered at e_0 with neighbors e_k, $k \in K \setminus \{i\}$, is given by

$$-\sum_{k \in K \setminus \{i\} : N(e_0 \cap e_k) \neq \emptyset} z_{u_k} + \sum_{v \in e_0 \setminus \bigcup_{k \in K \setminus \{i\}} e_k} z_v + \sum_{k \in K \setminus \{i\}} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{8}$$

where ω denotes the number of connected components of $\tilde{G}' = (e_0, \tilde{E}')$ and the sets $N(e_0 \cap e_k)$, $k \in K \setminus \{i\}$ are obtained according to the running intersection ordering $\mathcal{O}'$. Now consider the edge e_i and denote by u a node in $e_0 \cap e_i$. Then the following inequality is present in $\mathrm{MP}_G^{\mathrm{LP}}$:

$$-z_u + z_{e_i} \leq 0. \tag{9}$$

It is simple to see that $e_0 \setminus \cup_{k \in K} e_k = e_0 \setminus \cup_{k \in K \setminus \{i\}} e_k$. Moreover, the number of connected components of the two hypergraphs $\tilde{G} = (e_0, \tilde{E})$ and $\tilde{G}'$ are identical. In addition, by construction, the sets $N(e_0 \cap e_k)$, $k \in K \setminus \{i\}$, associated with $\mathcal{O}'$ coincide with those associated with $\mathcal{O}$. Finally, the set $N(e_0 \cap e_i)$ obtained using $\mathcal{O}$ is given by $N(e_0 \cap e_i) = e_0 \cap e_i$, since by assumption $e_0 \cap e_i \subseteq e_0 \cap e_j$ and $e_0 \cap e_j$ appears before $e_0 \cap e_i$. It then follows that the running intersection inequality under consideration can be obtained by adding inequalities (8) and (9).

Consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$. Assume that this inequality does not satisfy condition (ii), i.e. there exist $i \in K$ and $u \in V(G)$ such that $e_0 \cap e_i = \{u\}$. We can assume that the inequality satisfies condition (i); thus we have $u \notin e_k \cap e_0$ for every $k \in K \setminus \{i\}$. Consider a running intersection ordering $\mathcal{O}$ of $\tilde{E}$ defined by (3) and let the set $N(e_0 \cap e_k)$, $k \in K$, be defined by (2). It then follows that $N(e_0 \cap e_i) = \emptyset$ and that the sequence $\mathcal{O}'$ obtained by removing $e_0 \cap e_i$ from $\mathcal{O}$ is a running intersection ordering of the set $\tilde{E}' = \{e_0 \cap e_k : k \in K \setminus \{i\}\}$. In addition, the sets $N(e_0 \cap e_k)$, $k \in K \setminus \{i\}$, associated with $\mathcal{O}'$ are identical to those associated with $\mathcal{O}$. Hence, a running intersection inequality centered at e_0 with neighbors e_k, $k \in K \setminus \{i\}$ is given by

$$-\sum_{k \in K : N(e_0 \cap e_k) \neq \emptyset} z_{u_k} + \sum_{v \in e_0 \setminus \bigcup_{k \in K \setminus \{i\}} e_k} z_v + \sum_{k \in K \setminus \{i\}} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{10}$$

where ω denotes the number of connected components of the hypergraph $\tilde{G}' = (e_0, \tilde{E}')$. Now consider the edge e_i; clearly, the following inequality is present in $\mathrm{MP}_G^{\mathrm{LP}}$:

$$-z_u + z_{e_i} \leq 0. \tag{11}$$

It is simple to see that $e_0 \setminus \bigcup_{k \in K \setminus \{i\}} e_k = \{u\} \cup (e_0 \setminus \bigcup_{k \in K} e_k)$. In addition, the number of connected components of $\tilde{G} = (e_0, \tilde{E})$ and $\tilde{G}'$ are identical. It then follows that the running intersection inequality under consideration can be obtained by summing up inequalities (10) and (11).

Finally, consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$ that does not satisfy condition (iii); i.e., there exist $i, j \in K$ with $u_i, u_j \in N(e_0 \cap e_i) \cap N(e_0 \cap e_j)$ such that $u_i \neq u_j$. We now construct two other running intersection inequalities entered at e_0 with neighbors e_k, $k \in K$, for which we select the same node from each $N(e_0 \cap e_k)$, for all $k \in K \setminus \{i, j\}$ as the original inequality, but for first one we let $u'_i = u'_j = u_i$, while for the second one we let $u''_i = u''_j = u_j$. It is then simple to check that the running intersection inequality under consideration can be obtained by adding these two inequalities both of which are present in $\mathrm{MP}_G^{\mathrm{RI}}$. □

2.4 Facet-defining inequalities

We conclude this section by showing that under certain assumptions running intersection inequalities are facet-defining for their support hypergraphs. This in turn indicates their effectiveness in constructing stronger polyhedral relaxations for general multilinear polytopes whose hypergraphs contains support hypergraphs of running intersection inequalities as section hypergraphs.

Proposition 4. *Consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$, and let G denote its support hypergraph. Assume that the inequality satisfies the following conditions:*

(1) $|e_0 \cap e_k| \geq 2$ for all $k \in K$,

(2) for every $K' \subseteq K$ such that $e_0 \cap (\cap_{k \in K'} e_k) \neq \emptyset$ we have $e_0 \cap (e_i \setminus \cup_{k \in K' \setminus \{i\}} e_k) \neq \emptyset$ for all $i \in K'$,

(3) each nonempty $N(e_0 \cap e_k)$, $k \in K$, intersects the set $U := \{u_k : k \in K,\ N(e_0 \cap e_k) \neq \emptyset\}$ in only one node.

Then this running intersection inequality defines a facet of MP_G.

Proof. Consider a running intersection inequality defined by (4).We start by identifying a set of points in $\mathcal{S}_G$ that satisfy this inequality tightly. Subsequently, we show that any nontrivial valid inequality $az \leq \alpha$ for $\mathcal{S}_G$ that is satisfied tightly at all such points coincides with (4) up to a positive scaling. Since MP_G is full dimensional [11], this in turn implies that inequality (4) defines a facet of MP_G.

Let $\tilde{G} = (e_0, \tilde{E})$, where $\tilde{E}$ is given by (3). As in the proof of Proposition 1, we denote by $\tilde{G}_1, \ldots, \tilde{G}_\omega$ the connected components of $\tilde{G}$. Consider a partition of K given by $K = \bigcup_{i=1}^{\omega} K_i$, where K_i contains the indices of the edges of $\tilde{G}_i$. Let Ω contain those indices $i \in \{1, \ldots, \omega\}$ for which $K_i \neq \emptyset$. By Lemma 2, for each $i \in \Omega$ there exists a unique index r_i in K_i with $N(e_0 \cap e_{r_i}) = \emptyset$. Define

$$\gamma_{G_i} = - \sum_{k \in K_i \setminus \{r_i\}} z_{u_k} + \sum_{k \in K_i} z_{e_k}.$$

Then, it can be checked that

Claim 3. *Let $z \in \mathcal{S}_G$. Then:*

(i) If $z_{u_k} = 1$ for all $k \in K_i \setminus \{r_i\}$ and $z_{e_k} = 1$ for all $k \in K_i$, then $\gamma_{G_i} = 1$,

(ii) If $z_{u_k} = z_{e_k}$ for all $k \in K_i \setminus \{r_i\}$ and $z_{e_{r_i}} = 0$, then $\gamma_{G_i} = 0$.

For notational simplicity, in the following, let $V_0 = e_0 \setminus \cup_{k \in K} e_k$. To identify the tight points of (4), we consider two cases:

(I) $z_{e_0} = 1$: a point in $\mathcal{S}_G$ satisfies (4) tightly if and only if $\gamma_{G_i} = 1$ for all $i \in \Omega$.

(II) $z_{e_0} = 0$: a point in $\mathcal{S}_G$ satisfies (4) tightly if and only if one of the following is satisfied:

- (II′) $z_v = 1$ for all $v \in V_0$, $\gamma_{G_j} = 0$ for some $j \in \Omega$ and $\gamma_{G_i} = 1$ for all $i \in \Omega \setminus \{j\}$,
- (II″) $V_0 \neq \emptyset$, $z_w = 0$ for some $w \in V_0$, $z_v = 1$ for all $v \in V_0 \setminus \{w\}$, and $\gamma_{G_i} = 1$ for all $i \in \Omega$.

If $V_0 \neq \emptyset$, by part (i) of Claim 3, it is simple to check that substituting tight points of type (I) and (II″) in $az \leq \alpha$, yields

$$a_v + a_{e_0} = 0, \quad \forall v \in V_0. \tag{12}$$

Define $U_j = \cup_{k \in K_j \setminus \{r_j\}} u_k$ for all $j \in \Omega$. For each $j \in \Omega$ with $\cup_{k \in K_j} e_k \setminus U_j \neq \emptyset$, by part (ii) of Claim 3, we construct two tight points of type (II′) as follows: the first tight point is obtained by letting $z_v = 0$ for all $v \in \cup_{k \in K_j} e_k$. The second tight point is obtained by letting $z_w = 1$ for some $w \in (\cup_{k \in K_j} e_k) \setminus U_j$ and $z_v = 0$ for all $v \in \cup_{k \in K_j} e_k \setminus \{w\}$. Note that from condition (1) it follows that $e_0 \cap \cup_{k \in K_j} e_k \setminus \{w\} \neq \emptyset$. By construction, in both tight points we have $z_{u_k} = z_{e_k} = 0$ for all $k \in K_j \setminus \{r_j\}$ and $z_{e_{r_j}} = 0$. Substituting these two points in $az \leq \alpha$ and subtracting the resulting expressions, gives $a_w = 0$. Using a similar line of arguments for each $w \in (\cup_{k \in K_j} e_k) \setminus U_j$ and each $j \in \Omega$, we obtain

$$a_v = 0, \quad \forall v \in \bigcup_{k \in K} e_k \setminus \bigcup_{j \in \Omega} U_j. \tag{13}$$

Let e_ℓ denote a leaf of $E(\tilde{G}_j)$. We claim that $e_0 \cap e_\ell \setminus U_j$ is nonempty. If $U_j = \emptyset$, then the statement is trivial. Otherwise, by definition of a leaf $e_0 \cap e_\ell \setminus U_j \supseteq e_0 \cap e_\ell \setminus e_0 \cap e_h$ for some $h \in K_j$ such that $h \neq \ell$. Moreover, from condition (2) it follows that $e_0 \cap (e_\ell \setminus e_h) \neq \emptyset$. Now we construct two tight points as follows: the first point is a tight point of type (I). The second point is obtained by letting $z_w = 0$ for some $w \in e_0 \cap e_\ell \setminus U_j$ and $z_v = 1$ for all $v \in \cup_{k \in K_j} e_k \setminus \{w\}$. This point is a tight point of type (II$'$). To see this, consider a running intersection ordering of $\tilde{E}$ in which $e_0 \cap e_\ell$ is the first element. Note that by part (ii) of Lemma 1 such an ordering exists. It then follows that at this tight point we have $z_{u_k} = z_{e_k} = 1$ for all $k \in K_j \setminus \{\ell\}$ and $z_{e_\ell} = 0$. By (13), we have $a_w = 0$. Substituting these two points in $az \leq \alpha$ and subtracting the resulting relations we obtain

$$a_{e_k} + a_{e_0} = 0, \quad \forall k \in K : e_k \text{ is a leaf of } \tilde{E}. \tag{14}$$

Again consider a tight point of type (II$'$) in which $\gamma_{G_j} = 0$ for some $j \in \Omega$ by letting $z_{e_k} = 0$ for all $k \in K_j$ and $z_{u_k} = 0$ for all $k \in K_j \setminus \{r_j\}$. Consider a node w in the set U_j defined above. Denote by K' the index set of all edges in K_j with $e_k \supset w$. Let $\ell \in K'$ and consider a running intersection ordering of $\tilde{E}$ in which $e_0 \cap e_\ell$ is the first element. The existence of such an ordering follows from part (ii) of Lemma 1. Now, construct a second tight point of type (II$'$) in which we have $z_w = 1$. By condition (3), we have $u_k = w$ for all $k \in K' \setminus \{\ell\}$, as by construction, $N(e_0 \cap e_k) \supseteq w$ for all $k \in K' \setminus \{\ell\}$. Moreover, by condition (2), there exists a node $v_\ell \in e_0 \cap (e_\ell \setminus \cup_{k \in K' \setminus \{\ell\}} e_k)$. It then follows that by letting $z_{v_\ell} = 0$ and $z_w = 1$, we can construct a tight point of type (II$'$) in $\mathcal{S}_G$ such that $z_{e_\ell} = 0$, $z_{u_k} = z_{e_k} = 1$ for all $k \in K' \setminus \{\ell\}$ and $z_{u_k} = z_{e_k} = 0$ for all $k \in K_j \setminus K'$. Substituting these two points in $az \leq \alpha$ and using (13), yields

$$(|K'| - 1) a_w + \sum_{k \in K' \setminus \{\ell\}} a_{e_k} = 0, \quad \forall \ell \in K'.$$

It then follows that for each $w \in U$ we have

$$a_w + a_{e_k} = 0, \quad \forall k \in K \text{ such that } e_k \supset w. \tag{15}$$

Together with (14), this implies that

$$a_{e_k} + a_{e_0} = 0, \quad \forall k \in K. \tag{16}$$

Finally, by substituting the tight point of type (I) we get $\alpha = \sum_{p \in V \cup E} a_p$. Together with (12), (13), (15), (16), this implies that $az \leq \alpha$ coincides with inequality (4) up to a positive scaling, implying that (4) defines a facet of MP_G. □

In particular, Proposition 4 implies the following:

Corollary 2. *Consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$. Suppose that $|e_0 \cap e_k| \geq 2$ for all $k \in K$ and $|e_0 \cap e_k \cap e_{k'}| \leq 1$ for all $k, k' \in K$. Then this inequality defines a facet of the multilinear polytope of its support hypergraph.*

Proof. To prove the statement it suffices to show conditions (2) and (3) of Proposition 4 are satisfied. First consider condition (2); since $|e_0 \cap e_k \cap e_{k'}| \leq 1$ for all $k, k' \in K$, it follows that for any $K' \subseteq K$, the set $e_0 \cap (\cap_{k \in K'} e_k)$ consists of at most a single node. Moreover, if $e_0 \cap (\cap_{k \in K'} e_k) = \{v\}$, then $e_0 \cap e_k \cap e_{k'} = \{v\}$ for all $k, k' \in K'$. Hence, for each $i \in K'$ we have $e_0 \cap (e_i \setminus \cup_{k \in K' \setminus \{i\}} e_k) = (e_0 \cap e_i) \setminus \{v\}$, and the latter is nonempty as by assumption $|e_0 \cap e_i| \geq 2$. Condition (3) is satisfied as the assumption $|e_0 \cap e_k \cap e_{k'}| \leq 1$ for all $k, k' \in K$ implies that $|N(e_0 \cap e_k)| \leq 1$ for all $k \in K$. □

We should remark that the converse of Proposition 4 does not hold in general; namely, while by Proposition 3, condition (1) is necessary, one can construct facet-defining inequalities that do not satisfy conditions (2) and (3). In fact, in Example 1, inequalities (6) are facet-defining but they do not satisfy condition (3) of Proposition 4. We believe that a complete characterization for facetness of running intersection inequalities depends on the precise structure of the support hypergraph. Nonetheless, the aforementioned sufficient condition serves as theoretical justification for effectiveness of running intersection inequalities in constructing tighter relaxations for general multilinear sets. In the next section, we consider a class of acyclic hypergraphs for which the running intersection relaxation coincides with the multilinear polytope.

3 Convex hull characterizations

In this section we show that for hypergraphs with a certain degree of acyclicity the running intersection relaxation coincides with the multilinear polytope. To this end, we briefly review different types of cycles in hypergraphs.

3.1 Hypergraph acyclicity

Unlike graphs for which there is a single natural notion of acyclic graphs, there are several non-equivalent definitions of acyclicity for hypergraphs which collapse to graph acyclicity for the special case of ordinary graphs. In fact, the notion of graph acyclicity has been extended to several different degrees of acyclicity of hypergraphs [14]. Among the most widely-used ones one can cite, in increasing order of generality, Berge-acyclicity, γ-acyclicity, and β-acyclicity. Next, we briefly review these concepts as they play a crucial role in our subsequent developments (see [5] for an exposition).

A *Berge-cycle* in G of length t is a chain $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_{t+1}$ such that $v_{t+1} = v_1$ and $t \geq 2$. A γ*-cycle* in G is a Berge-cycle such that $t \geq 3$, and the node v_i belongs to e_{i-1}, e_i and no other e_j, for all $i = 2, \ldots, t$. A β*-cycle* in G is a γ-cycle such that the node v_1 belongs to e_1, e_t and no other e_j. A hypergraph is *Berge-acyclic* (resp. γ*-acyclic*, β*-acyclic*) if it does not contain any Berge-cycle (resp. γ-cycle, β-cycle). Throughout this paper, given any cycle $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$, we denote by $V(C) = \{v_1, \ldots, v_t\}$ the nodes of C, and by $E(C) = \{e_1, \ldots, e_t\}$ the edges of C.

Consider a hypergraph $G = (V, E)$ and let V' be a subset of V. We define the *subhypergraph* of G induced by V' as the hypergraph $G_{V'}$ with node set V' and with edge set $\{e \cap V' : e \in E, |e \cap V'| \geq 2\}$. For every edge e of $G_{\bar{V}}$, there may exist several edges e' of G satisfying $e = e' \cap \bar{V}$; we denote by $e'(e)$ one such arbitrary edge of G. For ease of notation, we often identify an edge e of $G_{\bar{V}}$ with an edge $e'(e)$ of G. Next, we present a couple of basic properties of β-acyclic hypergraphs that will be used to prove our main results.

Lemma 3. *Let $G = (V, E)$ be a hypergraph. If the subhypergraph $G_{V'}$ contains a β-cycle of length t, then G contains a β-cycle of length t. In particular, if G is β-acyclic, then $G_{V'}$ is β-acyclic as well.*

Proof. Suppose that $G_{V'}$ contains a β-cycle $v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$. It is simple to check that $v_1, e'(e_1), v_2, e'(e_2), \ldots, v_t, e'(e_t), v_1$ is a β-cycle in G. □

The following result which first appeared in [4] relates the concepts of β-acyclicity and running intersection property.

Lemma 4. *A hypergraph $G = (V, E)$ is β-acyclic if and only if every $E' \subseteq E$ has the running intersection property.*

3.2 Tightness of running intersection relaxation

In [12] we define the flower relaxation as the polytope obtained by adding all flower inequalities for a multilinear set to its standard linearization. Subsequently, we show that the flower relaxation is the multilinear polytope if and only if the underlying hypergraph is γ-acyclic. In the remainder of this paper, we study the sharpness of the running intersection relaxation. As we detail next, it suffices to limit attention to β-acyclic hypergraphs. Denote by R a *relaxation* of the Multilinear set; namely, R is a function that associates to each hypergraph G a set R_G containing all points in $\mathcal{S}_G$. Consider a hypergraph $G = (V, E)$ and let $\bar{V}$ be a subset of V. Define

$$L_{\bar{V}} := \{z \in \mathbb{R}^{V+E} : z_v = 1 \ \forall v \in V \setminus \bar{V}\}. \tag{17}$$

Denote by $\text{proj}_{G_{\bar{V}}}(R_G \cap L_{\bar{V}})$ the set obtained from $R_G \cap L_{\bar{V}}$ by projecting out all variables z_v, for all $v \in V \setminus \bar{V}$, and z_f, for all $f \in E \setminus \{e'(e) : e \in E(G_{\bar{V}})\}$. In [12] we show the following equivalence for the Multilinear polytope:

Lemma 5. *Let $G = (V, E)$ be a hypergraph and let the set $L_{\bar{V}}$ be defined by* (17) *for some $\bar{V} \subseteq V$. Then $MP_{G_{\bar{V}}} = \text{proj}_{G_{\bar{V}}}(MP_G \cap L_{\bar{V}})$.*

Next, we present a weaker version of the this result for the running intersection relaxation. We state this result without a proof, as the proof is a straightforward generalization of the proof of Lemma 13 in [12] wherein we show that a similar inclusion relation holds for the flower relaxation.

Lemma 6. *Let $G = (V, E)$ be a hypergraph and let the set $L_{\bar{V}}$ be defined by* (17) *for some $\bar{V} \subseteq V$. Then $MP^{RI}_{G_{\bar{V}}} \subseteq \text{proj}_{G_{\bar{V}}}(MP^{RI}_G \cap L_{\bar{V}})$.*

Proposition 5. *If the hypergraph G is not β-acyclic, then $MP_G \subset MP^{RI}_G$.*

Proof. Suppose that G contains at least one β-cycle. Denote by C a β-cycle of minimum length, say t. To show that $\text{MP}_G \subset \text{MP}^{\text{RI}}_G$, by Lemma 5 and Lemma 6, it is sufficient to prove that $\text{MP}_{G_{V(C)}} \subset \text{MP}^{\text{RI}}_{G_{V(C)}}$.

Define the set $\tilde{E} := \{e \cap V(C) : e \in E(C)\}$. Clearly, $\tilde{E} \subseteq E(G_{V(C)})$. First suppose that $\tilde{E} = E(G_{V(C)})$; i.e., $G_{V(C)}$ is a graph that consists of a chordless cycle. The inclusion $\text{MP}_{G_{V(C)}} \subset \text{MP}^{\text{RI}}_{G_{V(C)}}$ is then valid as the odd-cycle inequalities are facet-defining for $\text{MP}_{G_{V(C)}}$ [19] and are clearly not implied by $\text{MP}^{\text{RI}}_{G_{V(C)}}$.

Next, suppose that $\tilde{E} \subset E(G_{V(C)})$. Let $\bar{e}$ be in $E(G_{V(C)}) \setminus \tilde{E}$. We claim that $\bar{e} = V(C)$. To obtain a contradiction, suppose that $\bar{e} \subset V(C)$. Then it is simple to check that $G_{V(C)}$ contains a β-cycle of length t' with $t' < t$. By Lemma 3, also G contains a β-cycle of length t'. However, this contradicts the assumption that C is β-cycle of G of minimum length. Hence $\bar{e} = V(C)$. This shows that $E(G_{V(C)}) = \tilde{E} \cup V(C)$, i.e. the hypergraph $G_{V(C)}$ consists of a chordless cycle enclosed by the edge $\bar{e}$. Denote by $az \leq \alpha$ an odd-cycle inequality corresponding to the chordless cycle in $G_{V(C)}$. Suppose that $a_e = -1$ for $e \in M \subseteq E(C)$ such that $|M| = 2h+1$ for some $h \geq 1$. It can be checked that any inequality of the form $az + hz_{\bar{e}} \leq \alpha$ defines a facet of $\text{MP}_{G_{V(C)}}$. However, such inequalities are not present in $\text{MP}^{\text{RI}}_{G_{V(C)}}$. Consequently, if the hypergraph G contains a β-cycle, we have $\text{MP}_{G_{V(C)}} \subset \text{MP}^{\text{RI}}_{G_{V(C)}}$. $\square$

Henceforth, we assume that the hypergraph $G = (V, E)$ is β-acyclic. By Lemmata 1 and 3, given any edge $e_0 \in E$ and a collection of adjacent edges e_k, $k \in K$, the set $\{e_0 \cap e_k : k \in K\}$ has the running intersection property. Hence, the polytope MP^{RI}_G can be simply obtained by adding to $\mathrm{MP}_G^{\mathrm{LP}}$ all inequalities of the form (4) with any $e_0 \in E$ as the center edge and any collection of adjacent edges e_k, $k \in K$. The following example reveals that even for β-acyclic hypergraphs, the running intersection relaxation may not coincide with the multilinear polytope.

Example 2. *Consider the hypergraph G with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_{12}, e_{123}, e_{124}, e_{1234}\}$, where the edge e_I contains the nodes with indices in I. It is simple to check that G is β-acyclic. It can be shown that the inequality $-z_{12} + z_{123} + z_{124} - z_{1234} \leq 0$ defines a facet of MP_G and is not valid for the running intersection relaxation of $\mathcal{S}_G$.* $\diamondsuit$

More generally, it can be checked that β-acyclic hypergraphs can have dense facet-defining inequalities. By dense facets, we mean facets whose support hypergraph contains almost all edges of the original hypergraph. This is in major contrast with the support hypergraph of running intersection inequalities in which there exists a center edge that is adjacent to all other edges. In the following, we characterize a class of β-acyclic hypergraphs for which we have $\mathrm{MP}_G = \mathrm{MP}_G^{\mathrm{RI}}$. We believe that for general β-acyclic hypergraphs MP_G has a far more complicated facial structure than $\mathrm{MP}_G^{\mathrm{RI}}$.

3.3 β-acyclic hypergraphs with the simple intersection property

We now introduce a class of β-acyclic hypergraphs for which we will show that the running intersection relaxation coincides with the multilinear polytope. We start by defining the simple intersection property. A hypergraph $G = (V, E)$ has the *simple intersection property* if there exist no three edges $e_0, e_1, e_2 \in E$ such that $|e_0 \cap e_1 \cap e_2| \geq 2$, $(e_0 \cap e_1) \setminus e_2 \neq \emptyset$, and $(e_0 \cap e_2) \setminus e_1 \neq \emptyset$.

As we detail in the following, if G is a β-acyclic hypergraph with the simple intersection property, then the subhypergraph G_e of G induced by any edge $e \in E(G)$ has a particular structure that enables us to characterize MP_{G_e} using a lift-and-project technique. Let us first define an almost laminar hypergraph. A hypergraph $G = (V, E)$ is *almost laminar* if it is β-acyclic and for every two edges $e_1, e_2 \in E$, we have $|e_1 \cap e_2| \leq 1$, or $e_1 \subset e_2$, or $e_2 \subset e_1$. The following is the key connection between β-acyclic hypergraphs with the simple intersection property and almost laminar hypergraphs.

Observation 1. *Let G be a β-acyclic hypergraph with the simple intersection property, and let $e_0 \in E(G)$. Then the subhypergraph G_{e_0} of G induced by e_0 is an almost laminar hypergraph.*

Proof. Since G is β-acyclic, by Lemma 3, the subhypergraph G_{e_0} is β-acyclic as well. Assume by contradiction that G_{e_0} is not almost laminar. Then there exist two edges e_1, e_2 of G such that $|(e_0 \cap e_1) \cap (e_0 \cap e_2)| \geq 2$, $e_0 \cap e_1 \not\subset e_0 \cap e_2$, and $e_0 \cap e_2 \not\subset e_0 \cap e_1$. Then edges e_0, e_1, e_2 satisfy $|e_0 \cap e_1 \cap e_2| \geq 2$, $(e_0 \cap e_1) \setminus e_2 = (e_0 \cap e_1) \setminus (e_0 \cap e_2) \neq \emptyset$, and $(e_0 \cap e_2) \setminus e_1 = (e_0 \cap e_2) \setminus (e_0 \cap e_1) \neq \emptyset$. This contradicts the fact that G has the simple intersection property. $\square$

The running intersection inequalities (4) can be greatly simplified if G is a β-acyclic hypergraph with the simple intersection property. Consider a collection of edges e_0, e_k, $k \in K$, satisfying conditions (i) and (ii) of Proposition 3, i.e. $e_0 \cap e_k \not\subseteq e_0 \cap e_{k'}$ for any $k, k' \in K$, and $|e_0 \cap e_k| \geq 2$ for all $k \in K$. By construction, $\tilde{G} = (e_0, \tilde{E})$, where $\tilde{E}$ is defined by (3), is a partial hypergraph of the subhypergraph of G induced by e_0. Hence, by Observation 1, $\tilde{G}$ is almost laminar; it then follows that each set $N(e_0 \cap e_k)$, $k \in K$, as defined by (2) consists of at most a single node. For each node $v \in e_0$, denote by $\delta_K(v)$ the number of edges in e_k, $k \in K$, that contain v. Then there exists only one running intersection inequality centered at e_0 with neighbors e_k, $k \in K$, and it can be checked that this inequality is of the form

$$\sum_{v \in e_0} (1 - \delta_K(v)) z_v + \sum_{k \in K} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{18}$$

where as before ω denotes the number of connected components of $\tilde{G}$.

3.4 Statement of results

In this section we state the results that we need to establish that the multilinear polytope of β-acyclic hypergraphs with the simple intersection property coincides with the running intersection relaxation. To streamline the presentation the proofs are given in Section 4.

In Section 3.4.1 we characterize the multilinear polytope of almost laminar hypergraphs using a lift-and-project type technique. Subsequently, in Section 3.4.2 we present a sufficient condition under which a multilinear set is decomposable into a collection of simpler multilinear sets. In Section 3.4.3 we employ the results of Sections 3.4.1 and 3.4.2 to obtain our main result. More precisely, we show that in a lifted space, the multilinear polytope of a β-acyclic hypergraph G with the simple intersection property is representable as the intersection of a collection of multilinear polytopes of almost laminar hypergraphs. This in turn gives us a compact extended formulation for MP_G. Finally, in Section 3.4.4, by projecting out the extra variables we show that in the original space we have $\text{MP}_G = \text{MP}_G^{\text{RI}}$.

3.4.1 The multilinear polytope of almost laminar hypergraphs

Recall that a hypergraph G is laminar if for any two edges $e_1, e_2 \in E(G)$, we have $e_1 \subset e_2$ or $e_2 \subset e_1$ or $e_1 \cap e_2 = \emptyset$. It then follows that almost laminar hypergraphs subsume laminar hypergraphs. In fact, if an almost laminar hypergraph is γ-acyclic, then it is laminar. It is important to note that almost laminar hypergraphs contain γ-cycles in general. For example, the hypergraph G defined as $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_{12}, e_{23}, e_{34}, e_{1234}\}$ is almost laminar and it contains two γ-cycles $C_1 = v_1, e_{12}, v_2, e_{23}, v_3, e_{1234}, v_1$ and $C_2 = v_2, e_{23}, v_3, e_{34}, v_4, e_{1234}, v_2$.

In [12], we show that the subhypergraph induced by an edge of a γ-acyclic hypergraph is laminar. Subsequently, we characterize the multilinear polytope of laminar hypergraphs by leveraging on a fundamental result due to Conforti and Cornuéjols regarding the connection between integral polyhedra and balanced matrices [7]. Namely, we show that the constraint matrix corresponding to the facet description of the multilinear polytope of laminar hypergraphs is balanced. A similar proof technique is not applicable to almost laminar hypergraphs as the concept of balancedness is only defined for $0, \pm 1$ matrices; that is, such a technique can only be used if the constraint matrix corresponding to the facet description of the multilinear polytope only contains $0, \pm 1$ entries. However, for almost laminar hypergraphs, some facet-defining inequalities have general integer-valued coefficients. We employ a lift-and-project type argument to characterize the multilinear polytope of almost laminar hypergraphs, which is significantly more involved than our earlier proof for laminar hypergraphs.

To state the facet-description of MP_G for an almost laminar hypergraph $G = (V, E)$, we make use of the following notation. For each edge $e \in E$, define $I(e) := \{p \in V \cup E : p \subset e, p \not\subset e', \text{ for } e' \in E, e' \subset e\}$ and denote by $\omega(e)$ the number of connected components in the hypergraph $H_e = (e, I(e) \cap E)$. For each $v \in V$, let $\delta_e(v)$ denote the number of edges in H_e containing v. It is simple to show that $\omega(e) = \sum_{v \in e} (1 - \delta_e(v)) + |I(e) \cap E|$.

Theorem 1. *Let $G = (V, E)$ be an almost laminar hypergraph. Then MP_G is described by the* *following system:*

$$\begin{aligned}
z_v &\le 1 && \forall v \in V\\
-z_p &\le 0 && \forall p \in V \cup E \ s.t.\ p \not\subset f,\ \textit{for every } f \in E \qquad (19)\\
-z_p + z_e &\le 0 && \forall e \in E,\ \forall p \in I(e)\\
\sum_{v \in e}(1-\delta_e(v))z_v + \sum_{p \in I(e)\cap E} z_p - z_e &\le \omega(e) - 1 && \forall e \in E.
\end{aligned}$$

The proof of Theorem 1 is given in Section 4.1.

Consider the inequalities of system (19). Clearly, the first two sets are present in $\mathrm{MP}^{\mathrm{LP}}_G$. The third set is present in $\mathrm{MP}^{\mathrm{LP}}_G$ if p is a node, and is a running intersection inequality if p is an edge. Finally, for each $e \in E$, the last inequality is present in $\mathrm{MP}^{\mathrm{LP}}_G$ if $I(e) \subset V$ and by (18) is a running intersection inequality otherwise. Hence, we have the following characterization:

Corollary 3. *Let G be an almost laminar hypergraph. Then $MP_G = MP^{RI}_G$.*

It is important to note that for an almost laminar hypergraph G, the relaxation $\mathrm{MP}^{\mathrm{RI}}_G$ in general contains many more running intersection inequalities than system (19). More precisely, for each edge $e \in E(G)$, inequalities (19) contain at most two running intersection inequalities in which e is the center edge, while in the description of $\mathrm{MP}^{\mathrm{RI}}_G$, the number of running intersection inequalities (18) centered at e grows exponentially with the number of neighbors. In addition, it can be shown that all running intersection inequalities in system (19) are facet-defining whereas many of the running intersection inequalities present in $\mathrm{MP}^{\mathrm{RI}}_G$ are redundant and identifying such redundant inequalities is not simple in general. This compact representation is the key property which enables us to employ a lift-and-project technique to characterize the convex hull.

3.4.2 A sufficient condition for decomposability of multilinear sets

Given hypergraphs $G_\alpha = (V_\alpha, E_\alpha)$ and $G_\omega = (V_\omega, E_\omega)$ we denote by $G_\alpha \cap G_\omega$ the hypergraph $(V_\alpha \cap V_\omega, E_\alpha \cap E_\omega)$ and by $G_\alpha \cup G_\omega$ the hypergraph $(V_\alpha \cup V_\omega, E_\alpha \cup E_\omega)$. Let G be a hypergraph and let G_α, G_ω be section hypergraphs of G such that $G_\alpha \cup G_\omega = G$. We say that the set $\mathcal{S}_G$ is *decomposable into the sets $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$* if

$$\operatorname{conv} \mathcal{S}_G = \operatorname{conv} \bar{\mathcal{S}}_{G_\alpha} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_\omega},$$

where $\bar{\mathcal{S}}_{G_\alpha}$ (resp. $\bar{\mathcal{S}}_{G_\omega}$) is the set of all points in the space of $\mathcal{S}_G$ whose projection in the space defined by G_α (resp. G_ω) is $\mathcal{S}_{G_\alpha}$ (resp. $\mathcal{S}_{G_\omega}$).

In [10] and [12] we derive sufficient conditions for decomposability of multilinear sets. In [10] we show that $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$ if the hypergraph $G_\alpha \cap G_\omega$ is complete. In [12] we show that $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$ if $\bar{e} = V(G_\alpha) \cap V(G_\omega)$ is an edge of G and every edge that is only present in G_α either contains $\bar{e}$ or is disjoint from it. In particular, our decomposition result in [12] enables us to characterize multilinear polytopes for Berge-acyclic and γ-acyclic hypergraphs by showing that the corresponding multilinear sets are decomposable into a collection of simpler subsets whose convex hulls can be obtained directly.

Next, in Theorem 2, we provide a new sufficient condition for decomposability of multilinear sets. The setting considered in Theorem 2 is significantly more involved than the ones described above. Namely, the edges of G_α may only contain a subset of nodes in $V(G_\alpha) \cap V(G_\omega)$. The new result requires a significantly more involved proof as our earlier tools in [10] and [12] are not applicable to the current setting. More precisely, the key step in proving all these decomposition results is to show that a vector $(\hat{z}_\alpha, \hat{z}_\cap, \hat{z}_\omega)$ can be written as a convex combination of vectors in $\mathcal{S}_G$ if $(\hat{z}_\alpha, \hat{z}_\cap)$ can be written as a convex combination of vectors in $\mathcal{S}_{G_\alpha}$ and $(\hat{z}_\cap, \hat{z}_\omega)$ can be written as a convex combination of vectors in $\mathcal{S}_{G_\omega}$. To prove the decomposition results in [10] and [12] it is sufficient to consider vectors in $\mathcal{S}_G$ obtained by combining only one vector in $\mathcal{S}_{G_\alpha}$ with only one vector in $\mathcal{S}_{G_\omega}$. However, to prove Theorem 2 it seems no longer sufficient to consider vectors in $\mathcal{S}_G$ obtained by combining only one vector in $\mathcal{S}_{G_\alpha}$ with one vector in $\mathcal{S}_{G_\omega}$. To address this issue, we exploit the special structure of G_α and partition its edge set into k subsets based on the nodes in $V(G_\alpha) \cap V(G_\omega)$ to which they are connected. This allows us to combine one vector in $\mathcal{S}_{G_\omega}$ with k vectors in $\mathcal{S}_{G_\alpha}$ (one per each element of the partition) that coincide in certain components of $G_\alpha \cap G_\omega$ and obtain a vector in $\mathcal{S}_G$. Finally, we show that any vector $(\hat{z}_\alpha, \hat{z}_\cap, \hat{z}_\omega) \in \mathrm{MP}_G$ can be written as a convex combination of the obtained vectors in $\mathcal{S}_G$.

We now state our decomposition result. The proof is given in Section 4.2.

Theorem 2. *Let G be a hypergraph, and let G_α, G_ω be section hypergraphs of G such that $G_\alpha \cup G_\omega = G$. Denote by $\bar{p} := V(G_\alpha) \cap V(G_\omega)$. Suppose that $\bar{p} \in V(G) \cup E(G)$ and that G_α is almost laminar. Then the set $\mathcal{S}_G$ is decomposable into $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$.*

3.4.3 A compact extended formulation of MP_G

We now use the result of Theorem 2 to obtain a compact extended formulation for the multilinear polytope of β-acyclic hypergraphs with the simple intersection property. We say that an edge is *maximal* if it is not strictly contained in any other edge.

Consider a β-acyclic hypergraph $G = (V, E)$ with the simple intersection property. If V is an edge of G, then by Lemma 1, G is an almost laminar hypergraph and consequently by Corollary 3 we have $\mathrm{MP}_G = \mathrm{MP}^{\mathrm{RI}}_G$. Henceforth, suppose that G has at least two maximal edges. Denote by $\bar{E}$ the set of all maximal edges of G, and define $\kappa := |\bar{E}|$. Then by Lemma 4, there exists a running intersection ordering $\mathcal{O} = \bar{e}_1, \ldots, \bar{e}_\kappa$ of $\bar{E}$. Let the sets $N(\bar{e}_j)$, $j \in \{1, \ldots, \kappa\}$ be as defined in (2). We now construct the hypergraph $G^+ = (V, E^+)$ obtained from G by adding at most $\kappa - 1$ auxiliary edges to E, defined as follows:

$$E^+ := E \cup \{N(\bar{e}_j) : |N(\bar{e}_j)| \geq 2,\ j \in \{2, \ldots, \kappa\}\}. \tag{20}$$

The following theorem provides a polynomial-size extended formulation for MP_G which contains at most $|V| + 2|E|$ variables and $2(|V| + (r+1)|E|)$ inequalities, where r denotes the maximum cardinality of the edges of G.

Theorem 3. *Let $G = (V, E)$ be a β-acyclic hypergraph with the simple intersection property. Denote by $\bar{e}_i$, $i = 1, \ldots, \kappa$, the maximal edges of G. Consider the hypergraph $G^+ = (V, E^+)$, where E^+ is defined by (20), and denote by G^+_i, $i = 1, \ldots, \kappa$, the section hypergraph of G^+ induced by $\bar{e}_i$. Then G^+_i, $i \in \{1, \ldots, \kappa\}$, is an almost laminar hypergraph and*

$$MP_{G^+} = \bigcap_{i=1}^{\kappa} MP_{G^+_i}. \tag{21}$$

The proof of Theorem 3 is given in Section 4.3. In essence, via a recursive application of our decomposition result stated in Theorem 2, we show that $\mathcal{S}_{G^+}$ is decomposable into a collection to multilinear sets of almost laminar hypergraphs. In particular, Theorem 3 implies that we can optimize over MP_G in polynomial time.

3.4.4 The explicit characterization of MP_G

The facet description of each polytope $\mathrm{MP}_{G_i^+}$ in (21) is given by system (19) in Theorem 1. By projecting out the auxiliary variables z_e, $e \in E^+ \setminus E$, from the description of MP_{G^+}, using Fourier-Motzkin elimination, we obtain an explicit characterization for MP_G:

Theorem 4. *Let G be a β-acyclic hypergraph with the simple intersection property. Then $MP_G = MP_G^{RI}$.*

The proof of Theorem 4 is given in Section 4.4.

It is important to note that while Theorem 4 provides an explicit description of MP_G in the original space, the polytope $\mathrm{MP}_G^{\mathrm{RI}}$ may contain exponentially many facet-defining inequalities in general (see Example 2 in [12] in which we give a γ-acyclic hypergraph G for which the number of facets of MP_G is not bounded by a polynomial in $|V(G)|$ and $|E(G)|$).

We conclude this section by remarking that the converse of Theorem 4 is not correct, in general. Obtaining a complete characterization of β-acyclic hypergraphs for which the running intersection relaxation coincides with the multilinear polytope is a topic of future research. On the computational side, we plan to incorporate the proposed running intersection inequalities in a branch-and-cut framework to construct tighter polyhedral relaxations of general MINLPs containing a collection of multilinear subexpressions.

4 Proofs of main results

In this section we provide the proofs of the theorems stated in Section 3.

4.1 Proof of Theorem 1

Let $G = (V, E)$ be an almost laminar hypergraph. We prove the theorem by induction on the number of nodes of G. In the base case, G consists of a single node v. In this case, system (19) simplifies to $0 \leq z_v \leq 1$, which is clearly the multilinear polytope. To perform the inductive step, we a select a particular node $\tilde{v}$ in G. To do so, we first define an extremal element.

For each $e \in E$, define $I(e) := \{p \in V \cup E : p \subset e, p \not\subset e', \text{ for } e' \in E, e' \subset e\}$ and $U(e) := \{v \in V : \{v\} = e_1 \cap e_2, \text{ for some } e_1, e_2 \in I(e) \cap E\}$. Let $\hat{e} \in E$ and consider a partial hypergraph of G denoted by $H_{\hat{e}}$ with $V(H_{\hat{e}}) = \hat{e}$ and $E(H_{\hat{e}}) = I(\hat{e}) \cap E$. We refer to an element $p \in I(\hat{e})$ as an *extremal element* of $H_{\hat{e}}$ if the set $w_p = p \cap (\bigcup_{e \supseteq \hat{e}} U(e))$ is either empty or consists of a single node and $w_p \neq p$. If an extremal p is an edge, we refer to it as an *extremal-edge*. Since $p \subset \hat{e}$, it follows that $p \cap (\bigcup_{e \supset \hat{e}} U(e)) = p \cap (\hat{e} \cap (\bigcup_{e \supset \hat{e}} U(e))) = p \cap w_{\hat{e}}$. Hence, we have $w_p = (p \cap w_{\hat{e}}) \cup (p \cap U(\hat{e}))$. The hypergraph $H_{\hat{e}}$ is a partial hypergraph of the β-acyclic hypergraph G. Hence by part (i) of Lemma 1 and Lemma 4, the set $E(H_{\hat{e}})$ has at least two leaves. From the definition of $H_{\hat{e}}$ it follows that an edge $\tilde{e}$ is a leaf of $E(H_{\hat{e}})$ when the set $N(\tilde{e}) = \tilde{e} \cap (\cup_{e \in E(H_{\hat{e}}) \setminus \{\tilde{e}\}} e) = \tilde{e} \cap U(\hat{e})$ consists of at most one node. Since $N(\tilde{e}) \subseteq w_{\tilde{e}}$, it follows that every extremal-edge of $H_{\hat{e}}$ is a leaf of $E(H_{\hat{e}})$ but the converse is not true. In fact, $H_{\hat{e}}$ may not have any extremal-edges in general. However, as we show next, in the special case where $\hat{e}$ is already an extremal-edge, $H_{\hat{e}}$ has an extremal-edge as well.

Claim 4. *Let $e_j \in I(e_i)$ and suppose that e_j is an extremal-edge of H_{e_i}. If $I(e_j) \cap E \neq \emptyset$, then H_{e_j} has an extremal-edge.*

Proof of claim. We show that H_{e_j} has an extremal-edge e_k. We have $w_{e_k} = (e_k \cap w_{e_j}) \cup (e_k \cap U(e_j))$. Since e_j is an extremal-edge of H_{e_i}, the set w_{e_j} is either empty or consists of a single node. If H_{e_j} has a connected component consisting of a single edge e_k, then e_k is an extremal-edge of H_{e_j} as $e_k \cap U(e_j) = \emptyset$, implying $w_{e_k} \subseteq w_{e_j}$. Hence, suppose that each connected component in H_{e_j} has at least two edges. By part (i) of Lemma 1, the edge set of each connected component in H_{e_j} has at least two leaves e' and e''; that is, each of the two sets $e' \cap U(e_j)$ and $e'' \cap U(e_j)$ consist of a single node. Clearly, if (i) $w_{e_j} \subset e'$ and $w_{e_j} \subset e''$ which implies $w_{e_j} \subset U(e_j)$ or (ii) $w_{e_j} \not\subset e'$ and $w_{e_j} \not\subset e''$, then we have $w_{e'} = e' \cap U(e_j)$ and $w_{e''} = e'' \cap U(e_j)$, implying both e' and e'' are extremal-edges of H_{e_j}. Hence, the only remaining case is $w_{e_j} \subset e'$ and $w_{e_j} \not\subset e''$ (resp. $w_{e_j} \not\subset e'$ and $w_{e_j} \subset e''$), in which case e'' (resp. e') is an extremal-edge of H_{e_j}. Hence, H_{e_j} has an extremal-edge. ⋄

We now describe the algorithm to select the node $\tilde{v}$ for the inductive step. Without loss of generality, we assume that G has an edge containing all its nodes; i.e., $e_0 := V \in E$, as otherwise by Theorem 1 in [10], the multilinear set $\mathcal{S}_G$ is decomposable into a collection multilinear subsets each of which corresponds to an almost laminar hypergraph with an edge containing all of its nodes. First consider the edge e_0; if $I(e_0) = V$, we let $\tilde{v}$ be any node in e_0. Otherwise, by Claim 4, we select an extremal-edge of H_{e_0} denoted by e_1. If $I(e_1) \subset V$, then we let $\tilde{v}$ be a node in $e_1 \setminus w_{e_1}$. Otherwise, we apply Claim 4 recursively, until we obtain an extremal-edge e_t of $H_{e_{t-1}}$ with $I(e_t) \subset V$ and we let $\tilde{v} \in e_t \setminus w_{e_t}$. Note that $e_j \setminus w_{e_j} \neq \emptyset$ for all $j \in \{1, \ldots t\}$, as for the extremal edge e_j, the set w_{e_j} is either empty or consists of a single node. Denote by $\tilde{E}$ the set of all edges of G containing the node $\tilde{v}$. By the above construction, the set $\tilde{E}$ consists of a sequence of nested edges $e_0 \supset e_1 \supset \ldots \supset e_t$, where each e_i, $i \in \{1, \ldots, t\}$ is an extremal-edge of $H_{e_{i-1}}$.

The inductive step. Denote by G_0 (resp. G_1) the hypergraph corresponding to the face of MP_G with $z_{\tilde{v}} = 0$ (resp. $z_{\tilde{v}} = 1$). We have $\mathrm{MP}_G = \mathrm{conv}(\mathrm{MP}_{G_0} \cup \mathrm{MP}_{G_1})$. Clearly, both G_0 and G_1 are almost laminar hypergraphs and $|V(G_0)| = |V(G_1)| = |V(G)| - 1$. Hence, MP_{G_0} and MP_{G_1} can be obtained from the induction hypothesis.

Then MP_{G_0} is given by

$$
\begin{aligned}
& z_{\tilde{v}} = 0 && \\
& z_v \leq 1 && \forall v \in V \setminus \tilde{v} \\
& z_e = 0 && \forall e \in \tilde{E} \\
& -z_p \leq 0 && \forall p \in V \cup E \setminus \tilde{E}, p \not\subset f, f \in E \setminus \tilde{E} \\
& -z_p + z_e \leq 0 && \forall e \in E \setminus \tilde{E},\ \forall p \in I(e) \\
& \textstyle\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1 && \forall e \in E \setminus \tilde{E}.
\end{aligned}
\tag{22}
$$

Moreover, MP_{G_1} is given by

$$
\begin{aligned}
& z_{\tilde{v}} = 1 && \\
& z_v \leq 1 && \forall v \in V \setminus \tilde{v} \\
& z_e = z_{e \setminus \{\tilde{v}\}} && \forall e \in \tilde{E} : e \setminus \{\tilde{v}\} \in V \cup E \\
& -z_{e_0} \leq 0 && \\
& -z_p + z_e \leq 0 && \forall e \in E,\ \forall p \in I(e) \\
& \textstyle\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1 && \forall e \in E.
\end{aligned}
\tag{23}
$$

The last inequalities of systems (22) and (23) follow from the facts that for each $e \in E$, we have $\delta_e(v) = 0$ for all $v \in I(e)$ and $\delta_e(v) = 1$ for all $v \in e \setminus \{U(e) \cup I(e)\}$. Using Balas formulation for the union of polytopes [1], it follows that the polytope MP_G is the projection onto the space of the

z variables of the polyhedron defined by the following system

$$
\begin{array}{ll}
z_p = z^0_p + z^1_p & \forall p \in V \cup E \\
z^0_{\tilde{v}} = 0 & \\
z^0_v \leq \lambda_0 & \forall v \in V \setminus \tilde{v} \\
z^0_e = 0 & \forall e \in \tilde{E} \\
-z^0_p \leq 0 & \forall p \in V \cup E \setminus \tilde{E}, p \not\subset f, f \in E \setminus \tilde{E} \\
-z^0_p + z^0_e \leq 0 & \forall e \in E \setminus \tilde{E},\ \forall p \in I(e) \\
\sum_{v \in U(e)} (1-\delta_e(v)) z^0_v + \sum_{p \in I(e)} z^0_p - z^0_e \leq (\omega(e)-1)\lambda_0 & \forall e \in E \setminus \tilde{E} \\
z^1_{\tilde{v}} = \lambda_1 & \\
z^1_v \leq \lambda_1 & \forall v \in V \setminus \tilde{v} \\
z^1_e = z^1_{e \setminus \{\tilde{v}\}} & \forall e \in \tilde{E} : e \setminus \{\tilde{v}\} \in V \cup E \\
-z^1_{e_0} \leq 0 & \\
-z^1_p + z^1_e \leq 0 & \forall e \in E,\ \forall p \in I(e) \\
\sum_{v \in U(e)} (1-\delta_e(v)) z^1_v + \sum_{p \in I(e)} z^1_p - z^1_e \leq (\omega(e)-1)\lambda_1 & \forall e \in E \\
\lambda_0 + \lambda_1 = 1 & \\
\lambda_0, \lambda_1 \geq 0\} &
\end{array} \tag{24}
$$

We now project out the variables z^0, z^1, λ_0, λ_1 from system (24) and obtain an explicit description for MP_G. From (24), it follows that $z^0_{\tilde{v}} = 0$, $z^1_{\tilde{v}} = z_{\tilde{v}}$, $\lambda_0 = 1 - z_{\tilde{v}}$, $\lambda_1 = z_{\tilde{v}}$, $z^0_v = z_v - z^1_v$ for all $v \in V \setminus \{\tilde{v}\}$, $z^1_e = z_e$, for all $e \in \tilde{E}$, $z^1_{e \setminus \{\tilde{v}\}} = z_e$ for all $e \in \tilde{E}$ such that $e \setminus \{\tilde{v}\} \in V \cup E$, $z^0_e = z_e - z^1_e$ for all $e \in E \setminus \tilde{E}$. Hence, by projecting out λ_0, λ_1, z^0_p for all $p \in V \cup E$ and z^1_p for all $p \in \{\tilde{v}\} \cup \tilde{E}$, we obtain:

$$
\begin{array}{lll}
z_v - z^1_v \leq 1 - z_{\tilde{v}} & & \forall v \in V \setminus \tilde{v} \\
-(z_p - z^1_p) \leq 0 & & \forall p \in I(e), e \in \tilde{E} \\
-(z_p - z^1_p) + (z_e - z^1_e) \leq 0 & & \forall e \in E \setminus \tilde{E},\ \forall p \in I(e) \\
\sum_{v \in U(e)} (1-\delta_e(v))(z_v - z^1_v) + \sum_{p \in I(e)} (z_p - z^1_p) - (z_e - z^1_e) & \leq (\omega(e)-1)(1 - z_{\tilde{v}}) & \forall e \in E \setminus \tilde{E}
\end{array} \tag{25}
$$

and

$$
\begin{array}{ll}
-z_{e_0} \leq 0 & \\
z^1_v \leq z_{\tilde{v}} & \forall v \in V \setminus \tilde{v} \\
-z^1_p + z^1_e \leq 0 & \forall e \in E,\ \forall p \in I(e) \\
\sum_{v \in U(e)} (1-\delta_e(v)) z^1_v + \sum_{p \in I(e)} z^1_p - z^1_e \leq (\omega(e)-1) z_{\tilde{v}} & \forall e \in E.
\end{array} \tag{26}
$$

In the following, we project out z^1_v, $v \in V \setminus \tilde{v}$, z^1_e, $e \in E \setminus \tilde{E}$ from systems (25) and (26) in a specific order, and show that the projection is given by (19).

Projection orderings for $I(e)$**.** For any $e \in E$, the elements of $I(e)$ have the running intersection property. To see this, note that the set of edges in $I(e)$ is a subset of the edge set of a β-acyclic hypergraph G, and hence by Lemma 4 has the running intersection property. In addition, by construction, the nodes in $I(e)$ are not contained in any edge in $I(e)$. Now suppose that e is an extremal-edge of H_f, where $e \in I(f)$. Let p_s be an element of $I(e)$ that contains w_e. Clearly, if $w_e = \emptyset$, then p_s can be any element of $I(e)$. We define a *projection ordering for* $I(e)$, denoted by $\bar{\mathcal{O}}(e)$, as a running intersection ordering of $I(e)$ in which p_s is the first element. By part (ii) of Lemma 1 such an ordering exists. We define the hypergraph (V', E') obtained from H_e by *removing* some $p \in I(e)$ as $V' := V(H_e) \setminus \{v : v \in p\}$ and $E' := E(H_e) \setminus \{p\}$. For any $p \in I(e)$, we denote by $H^{\leq p}_e$, the hypergraph obtained from H_e by removing all elements appearing after p in $\bar{\mathcal{O}}(e)$. By definition of $\bar{\mathcal{O}}(e)$ and the proof of Claim 4 we have:

Claim 5. *Let e be an extremal-edge of H_f, where $e \in I(f)$ and let $\bar{\mathcal{O}}(e) = p_1, \ldots p_r$, where $r = |I(e)|$, be a projection ordering for $I(e)$. Then p_j is an extremal element of $H_e^{\leq p_j}$ for all $j \in \{1, \ldots, r\}$.*

Consider the projection ordering $\bar{\mathcal{O}}(e)$ as defined in Claim (5). Define $U^{\leq p_j}(e) := \{v \in V : \{v\} = e_1 \cap e_2, e_1, e_2 \in E(H_e^{\leq p_j})\}$ and $\bar{w}_{p_j} := (p_j \cap w_e) \cup (p_j \cap U^{\leq p_j}(e))$. By definition of a projection ordering $\bar{\mathcal{O}}(e)$ we have

$$\bar{w}_{p_1} = w_e, \quad \bar{w}_{p_j} = N(p_j), \quad \forall 2 \leq j \leq r, \tag{27}$$

where the sets $N(p_j)$ are as defined in (2). Since e is an extremal-edge of H_f and $p_1, \ldots p_r$ is a running intersection ordering of $I(e)$, it is simple to see that $\bar{w}_{p_j}$ is either empty or consist of a single node. In the remainder of the proof, given an edge $e \in E$, we use a projection ordering $\bar{\mathcal{O}}(e) = p_1, \ldots, p_r$ to recursively project out variables z_{p_j}, $j \in \{1, \ldots, r\}$.

Projecting out z^1_p corresponding to G_e for some $e \in E \setminus \tilde{E}$. Consider an edge $\bar{e} \in E \setminus \tilde{E}$ and let $G_{\bar{e}}$ denote the section hypergraph of G induced by $\bar{e}$. Note that for an almost laminar hypergraph, the section hypergraph induced by an edge coincides with the subhypergraph induced by the same edge. Suppose that $\bar{e}$ is an extremal-edge of H_f, where $\bar{e} \in I(f)$. Our objective is to project out variables z^1_v for all $v \in V(G_{\bar{e}}) \setminus w_{\bar{e}}$ and z^1_e for all $e \in E(G_{\bar{e}}) \setminus \{\bar{e}\}$ from systems (25) and (26). To this end, we make use of the following result:

Claim 6. *Let $e \in E \setminus \tilde{E}$ and suppose that e is an extremal-edge of H_f, where $e \in I(f)$. Let $\bar{\mathcal{O}}(e)$ be a projection ordering for $I(e)$ with the corresponding sets $\bar{w}_p$, $p \in I(e)$ as defined by (27). Consider the following inequalities:*

$$\begin{cases} z^1_p \leq z_{\tilde{v}} & \text{if } \bar{w}_p = \emptyset, \text{ or } p = w_e, \ \forall p \in I(e) \\ z^1_p \leq z^1_{v_p}, \ z^1_{v_p} \leq z_{\tilde{v}} & \text{if } \bar{w}_p = \{v_p\}, \ \forall p \in I(e) \end{cases} \tag{28}$$

$$z^1_e \leq z^1_p \quad \forall p \in I(e) \tag{29}$$

$$\sum_{v \in U(e)} (1 - \delta_e(v)) z^1_v + \sum_{p \in I(e)} z^1_p - z^1_e \leq (\omega(e) - 1) z_{\tilde{v}} \tag{30}$$

$$\begin{cases} z_p - z^1_p \leq 1 - z_{\tilde{v}} & \text{if } \bar{w}_p = \emptyset, \text{ or } p = w_e, \ \forall p \in I(e) \\ z_p - z^1_p \leq z_{v_p} - z^1_{v_p}, \ z_{v_p} - z^1_{v_p} \leq 1 - z_{\tilde{v}} & \text{if } \bar{w}_p = \{v_p\}, \ \forall p \in I(e) \end{cases} \tag{31}$$

$$z_e - z^1_e \leq z_p - z^1_p \qquad \forall p \in I(e) \tag{32}$$

$$\sum_{v \in U(e)} (1 - \delta_e(v))(z_v - z^1_v) + \sum_{p \in I(e)} (z_p - z^1_p) - (z_e - z^1_e) \leq (\omega(e) - 1)(1 - z_{\tilde{v}}). \tag{33}$$

Then by projecting out z^1_p for all $p \in I(e) \cup U(e) \setminus w_e$, we obtain

$$\begin{array}{ll} z_p \leq 1 & \forall p \in U(e) \text{ and } \forall p \in I(e) \text{ s.t. } \bar{w}_p = \emptyset \\ z_p \leq z_{v_p} & \forall p \in I(e) \text{ s.t. } \bar{w}_p = \{v_p\} \\ z_e \leq z_p, & \forall p \in I(e) \\ \sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1 & \end{array} \tag{34}$$

together with

$$\begin{array}{l} z^1_e \leq z_{\tilde{v}} \\ z_e - z^1_e \leq 1 - z_{\tilde{v}}, \end{array} \tag{35}$$

if $w_e = \emptyset$, *and*

$$\begin{aligned} & z^1_e \leq z^1_{v_e} \\ & z_e - z^1_e \leq z_{v_e} - z^1_{v_e} \\ & z^1_{v_e} \leq z_{\tilde{v}} \\ & z_{v_e} - z^1_{v_e} \leq 1 - z_{\tilde{v}}, \end{aligned} \tag{36}$$

if $w_e = \{v_e\}$.

Proof of claim. First suppose that $w_e = \emptyset$. Let $\bar{p}$ be the last element of $\bar{\mathcal{O}}(e)$. We project out the variable $z^1_{\bar{p}}$ from inequalities (28)-(33) using Fourier-Motzkin elimination. From (28) and (31) we obtain

$$\begin{cases} z_{\bar{p}} \leq 1 & \text{if } \bar{w}_{\bar{p}} = \emptyset \\ z_{\bar{p}} \leq z_{v_{\bar{p}}} & \text{if } \bar{w}_{\bar{p}} = \{v_{\bar{p}}\}, \end{cases} \tag{37}$$

while from (29) and (32) we obtain

$$z_e \leq z_{\bar{p}}. \tag{38}$$

From (30) and (33) we obtain

$$\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1. \tag{39}$$

From (29) and (30) we obtain

$$\sum_{v \in U(e)} (1 - \delta_e(v)) z^1_v + \sum_{p \in I(e) \setminus \{\bar{p}\}} z^1_p \leq (\omega(e) - 1) z_{\tilde{v}}. \tag{40}$$

We claim that inequality (40) is redundant. To see this, consider a running intersection ordering $\mathcal{O}$ of $I(e)$ in which $\bar{p}$ is the first element. Note that by part (ii) of Lemma 1 such an ordering exists. Let the sets $N(p)$, $p \in I(e)$ be defined by (2). Now for each $p \in \mathcal{O} \setminus \{\bar{p}\}$, consider the following inequalities all of which are either present in system (26) or are implied by it: $z^1_p \leq z_{\tilde{v}}$ if $N(p) = \emptyset$, and $z^1_p \leq z^1_{v_p}$ if $N(p) = \{v_p\}$. By summing up these inequalities for all $p \in \mathcal{O} \setminus \{\bar{p}\}$, we obtain (40). By symmetry, projecting out $z^1_{\bar{p}}$ from (32) and (33) yields a redundant inequality. By projecting out $z^1_{\bar{p}}$ from (28) and (29) we obtain

$$z^1_e \leq z_{\tilde{v}}, \tag{41}$$

if $\bar{w}_{\bar{p}} = \emptyset$, and $z^1_e \leq z^1_{v_{\bar{p}}}$ if $\bar{w}_{\bar{p}} = \{v_{\bar{p}}\}$. The latter inequality is redundant as it is implied by inequalities (28), for some $p \neq \bar{p}$ such that $p \supset v_{\bar{p}}$. By symmetry, from (31) and (32) we obtain

$$z_e - z^1_e \leq 1 - z_{\tilde{v}} \tag{42}$$

if $\bar{w}_{\bar{p}} = \emptyset$, and we obtain a redundant inequality if $\bar{w}_{\bar{p}} = \{v_{\bar{p}}\}$. From (30) and (31) we obtain

$$\begin{cases} \sum_{v \in U(e)} (1 - \delta_e(v)) z^1_v + z_{\bar{p}} + \sum_{p \in I(e) \setminus \{\bar{p}\}} z^1_p - z^1_e \leq (\omega(e) - 2) z_{\tilde{v}} + 1 & \text{if } \bar{w}_{\bar{p}} = \emptyset \\ (2 - \delta_e(v_{\bar{p}})) z^1_{v_{\bar{p}}} - z_{v_{\bar{p}}} + \sum_{v \in U(e) \setminus \{v_{\bar{p}}\}} (1 - \delta_e(v)) z^1_v + z_{\bar{p}} + \sum_{p \in I(e) \setminus \{\bar{p}\}} z^1_p - z^1_e \leq & \\ \leq (\omega(e) - 1) z_{\tilde{v}} & \text{if } \bar{w}_{\bar{p}} = \{v_{\bar{p}}\} \end{cases} \tag{43}$$

Finally, the inequalities obtained by projecting out $z^1_{\bar{p}}$ from (28) and (33) are given by

$$\begin{cases} \sum_{v \in U(e)} (1 - \delta_e(v))(z_v - z^1_v) + z_{\bar{p}} + \sum_{p \in I(e) \setminus \{\bar{p}\}} (z_p - z^1_p) - (z_e - z^1_e) \leq & \\ \leq (\omega(e) - 2)(1 - z_{\tilde{v}}) + 1 & \text{if } \bar{w}_{\bar{p}} = \emptyset \\ (2 - \delta_e(v_{\bar{p}}))(z_{v_{\bar{p}}} - z^1_{v_{\bar{p}}}) - z_{v_{\bar{p}}} + \sum_{v \in U(e) \setminus \{v_{\bar{p}}\}} (1 - \delta_e(v))(z_v - z^1_v) + z_{\bar{p}} + & \\ + \sum_{p \in I(e) \setminus \{\bar{p}\}} (z_p - z^1_p) - (z_e - z^1_e) \leq (\omega(e) - 1)(1 - z_{\tilde{v}}) & \text{if } \bar{w}_{\bar{p}} = \{v_{\bar{p}}\} \end{cases} \tag{44}$$

Hence, projecting out $z^1_{\bar{p}}$ from inequalities (28)-(33), yields inequalities (37), (38), (39), (41), (42), (43) and (44). Denote by $\tilde{p}$ the element before $\bar{p}$ in $\bar{\mathcal{O}}(e)$. Clearly, among the inequalities obtained as a result of the above projection, the only ones containing $z^1_{\tilde{p}}$ are inequalities (43) and (44). Hence, to project out $z^1_{\tilde{p}}$ from the system (28)-(33), it suffices to consider inequalities (43) and (44) together with inequalities (28), (29), (31), and (32), for $p = \tilde{p}$. Using a similar line of arguments as above, it follows that the only non-redundant inequalities obtained from this projection are of the form (37) and (38) with $\bar{p}$ replaced by $\tilde{p}$ together with those obtained by projecting out $z^1_{\tilde{p}}$ from inequalities (31) (resp.(28)) and (43) (resp. (44)).

We now apply this approach recursively to project out z^1_p for all elements $p \in \bar{\mathcal{O}}(e)$ in reverse order. From (43) and (44) it follows that for a node $\bar{v} \in U(e)$, after projecting out z^1_p corresponding to the $\delta_e(\bar{v}) - 1$ edges with $\bar{w}_p = \{\bar{v}\}$, the coefficient of $z^1_{\bar{v}}$ in these inequalities becomes zero. Moreover, at this point, the only inequalities containing $z^1_{\bar{v}}$ are $z^1_{\bar{v}} \leq z_{\tilde{v}}$ and $z_{\bar{v}} - z^1_{\bar{v}} \leq 1 - z_{\tilde{v}}$. Hence, projecting out $z^1_{\bar{v}}$ yields $z_{\bar{v}} \leq 1$. As the number of elements p in $\bar{\mathcal{O}}(e)$ with $\bar{w}_p = \emptyset$ is equal to $\omega(e)$, after projecting out z^1_p for all $p \in \bar{\mathcal{O}}(e)$ from inequalities (31) and (43) we obtain $\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z^1_e \leq -z_{\tilde{v}} + \omega(e)$. However, this inequality is implied by inequalities (39) and (42). By symmetry, we conclude that the inequality obtained from the recursive projection of z^1_p, $p \in \bar{\mathcal{O}}(e)$ from (28) and (44) is redundant. Hence, by projecting out z^1_p, for all $p \in I(e) \cup U(e)$ from inequalities (28)-(33), we obtain inequalities (34) and (35).

Next, suppose that $w_e = \{v_e\}$ for some $v_e \in V$. Denote by p_s the first element in $\bar{\mathcal{O}}(e)$. Recall that by definition of $\bar{\mathcal{O}}(e)$, we have $p_s = v_e$ if $v_e \in I(e)$ and $p_s = \tilde{e}$ where $\tilde{e} \supset v_e$ is an edge in $I(e)$, otherwise. We employ the recursive projection as detailed above to project out z^1_p for all $p \in U(e) \cup I(e) \setminus \{p_s\}$. It then follows that projecting out z^1_p for all $p \in U(e) \cup I(e) \setminus \{p_s\}$ from inequalities (30) and (31) yields $\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e) \setminus \{p_s\}} z_p + z^1_{p_s} - z^1_e \leq \omega(e) - 1$. However, this inequality is implied by inequality (32) for $p = p_s$ and inequality (39). Symmetrically, we conclude that the inequality obtained by projecting out z^1_p for all $p \in U(e) \cup I(e) \setminus \{p_s\}$ from inequalities (28) and (33) is redundant. Finally, if $p_s = \tilde{e}$, we project out $z^1_{p_s}$, which is only present in inequalities (28), (29), (31), (32) with $p = \tilde{e}$ and $w_p = \{v_e\}$, implying its projection yields inequalities (36). Hence, we have shown that the final projection is given by inequalities (34) and (36). ◇

Recall that our objective is to project out z^1_v for all $v \in V(G_{\bar{e}}) \setminus w_{\bar{e}}$ and z^1_e for all $e \in E(G_{\bar{e}}) \setminus \{\bar{e}\}$ from systems (25) and (26), where $G_{\bar{e}}$ is the section hypergraph of G induced by $\bar{e}$ and $\bar{e} \in E \setminus \tilde{E}$ is an extremal-edge of H_f and $\bar{e} \in I(f)$. More precisely, we consider the following inequalities:

$$
\begin{aligned}
& z_v - z^1_v \leq 1 - z_{\tilde{v}} && \forall v \in \bar{e} \\
& -(z_p - z^1_p) + (z_e - z^1_e) \leq 0 && \forall e \in E(G_{\bar{e}}),\ \forall p \in I(e) \\
& \textstyle\sum_{v \in U(e)} (1 - \delta_e(v))(z_v - z^1_v) + \sum_{p \in I(e)} (z_p - z^1_p) - (z_e - z^1_e) \leq (\omega(e) - 1)(1 - z_{\tilde{v}}) && \forall e \in E(G_{\bar{e}}),
\end{aligned} \tag{45}
$$

and

$$
\begin{aligned}
& z^1_v \leq z_{\tilde{v}} && \forall v \in \bar{e} \\
& -z^1_p + z^1_e \leq 0 && \forall e \in E(G_{\bar{e}}),\ \forall p \in I(e) \\
& \textstyle\sum_{v \in U(e)} (1 - \delta_e(v)) z^1_v + \sum_{p \in I(e)} z^1_p - z^1_e \leq (\omega(e) - 1) z_{\tilde{v}} && \forall e \in E(G_{\bar{e}}).
\end{aligned} \tag{46}
$$

Claim 7. *Consider the section hypergraph $G_{\bar{e}}$ as defined above. By projecting out $z^1_v, v \in V(G_{\bar{e}}) \setminus w_{\bar{e}}$ and $z^1_e, e \in E(G_{\bar{e}}) \setminus \{\bar{e}\}$ from inequalities (45) and (46), we obtain*

$$
\begin{aligned}
& z_v \leq 1 && \forall v \in \bar{e} \\
& -z_p + z_e \leq 0 && \forall e \in E(G_{\bar{e}}),\ \forall p \in I(e) \\
& \textstyle\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1 && \forall e \in E(G_{\bar{e}}).
\end{aligned} \tag{47}
$$

together with

$$\begin{aligned} z_{\bar{e}}^1 &\leq z_{\tilde{v}} \\ z_{\bar{e}} - z_{\bar{e}}^1 &\leq 1 - z_{\tilde{v}}, \end{aligned} \tag{48}$$

if $w_{\bar{e}} = \emptyset$ *and*

$$\begin{aligned} z_{\bar{e}}^1 &\leq z_{v_{\bar{e}}}^1 \\ z_{\bar{e}} - z_{\bar{e}}^1 &\leq z_{v_{\bar{e}}} - z_{v_{\bar{e}}}^1 \\ z_{v_{\bar{e}}}^1 &\leq z_{\tilde{v}} \\ z_{v_{\bar{e}}} - z_{v_{\bar{e}}}^1 &\leq 1 - z_{\tilde{v}}, \end{aligned} \tag{49}$$

if $w_{\bar{e}} = \{v_{\bar{e}}\}$.

Proof of claim. The proof is by induction on the number of edges of $G_{\bar{e}}$. In the base case, we have $|E(G_{\bar{e}})| = 1$, implying $I(\bar{e}) \subset V$ and $U(\bar{e}) = \emptyset$. In this case, inequalities (45) and (46) coincide with inequalities (28)-(33) of Claim 6, by letting $e = \bar{e}$, in which case we have $\bar{w}_p = \emptyset$ for all $p \in \bar{\mathcal{O}}(\bar{e}) \setminus w_{\bar{e}}$. Hence, by projecting out z_p^1 for all $p \in I(\bar{e}) \setminus w_{\bar{e}}$, we obtain inequalities (34) and (35) (resp. (34) and (36)) which coincide with inequalities (47) and (48) (resp. (47) and (49)) for $w_{\bar{e}} = \emptyset$ (resp. $w_{\bar{e}} = \{v_{\bar{e}}\}$).

Suppose that $|E(G_{\bar{e}})| \geq 2$. Since $\bar{e}$ is an extremal-edge of H_f, where $\bar{e} \in I(f)$, we can construct a projection ordering $\bar{\mathcal{O}}(\bar{e})$ of $I(\bar{e})$ with the corresponding sets $\bar{w}_p$ defined by (27). Define $\tilde{\mathcal{O}}(\bar{e}) = \bar{\mathcal{O}}(\bar{e}) \setminus V(G_{\bar{e}})$ and let $r := |\tilde{\mathcal{O}}(\bar{e})|$. Denote by p_r the last element in $\tilde{\mathcal{O}}(\bar{e})$ and let G_{p_r} denote the section hypergraph of $G_{\bar{e}}$ induced by p_r. Clearly, G_{p_r} has at least one fewer edge than $G_{\bar{e}}$ and by construction p_r is an extremal-edge of $H_{\bar{e}}$. Hence, by the induction hypothesis, by projecting out z_v^1 for all $v \in V(G_{p_r}) \setminus \bar{w}_{p_r}$ and z_e^1 for all $e \in E(G_{p_r}) \setminus \{p_r\}$ from inequalities (45) and (46), we obtain the system defined in the statement of the claim with $\bar{e}$ replaced by p_r. Similarly, we consider in reverse order, each element $p_j \in \tilde{\mathcal{O}}(\bar{e})$ and since by Claim 5, p_j is an extremal-edge of $H_{\bar{e}}^{\leq p_j}$, we can use the induction hypothesis to project out z_v^1, $v \in V(G_{p_j}) \setminus \bar{w}_{p_j}$ and z_e^1, $e \in E(G_{p_j}) \setminus \{p_j\}$ from inequalities (45) and (46). It then follows that the remaining inequalities containing z_p^1, $p \in I(\bar{e}) \cup U(\bar{e})$ are identical to inequalities (28)-(33) defined in Claim 6 with $e = \bar{e}$; hence the final projection can be obtained accordingly and this completes the proof. ◇

Projecting out z_p^1 corresponding to G_e for some $e \in \tilde{E}$. Let $e \in \tilde{E}$ and denote by $\tilde{p}$ the element of $I(e)$ containing the node $\tilde{v}$. Consider a projection ordering $\bar{\mathcal{O}}(e)$ of $I(e)$ in which $\tilde{p}$ is the first element and as before, let the sets $\bar{w}_p$, $p \in \bar{\mathcal{O}}(e)$ be given by (27). Clearly, $z_e^1 = z_e$ and $z_{\tilde{p}}^1 = z_{\tilde{p}}$. Consider the following inequalities:

$$\begin{aligned}
&-z_p + z_p^1 \leq 0 \quad \forall p \in I(e) \setminus \{\tilde{p}\} \\
&\begin{cases} z_p - z_p^1 \leq 1 - z_{\tilde{v}} & \text{if } \bar{w}_p = \emptyset, \ \forall p \in I(e) \setminus \{\tilde{p}\} \\ z_p - z_p^1 \leq z_{v_p} - z_{v_p}^1, \ z_{v_p} - z_{v_p}^1 \leq 1 - z_{\tilde{v}} & \text{if } \bar{w}_p = \{v_p\}, \ \forall p \in I(e) \setminus \{\tilde{p}\} \end{cases} \\
&z_e \leq z_p^1 \quad \forall p \in I(e) \setminus \{\tilde{p}\} \\
&\begin{cases} z_p^1 \leq z_{\tilde{v}} & \text{if } \bar{w}_p = \emptyset, \ \forall p \in I(e) \setminus \{\tilde{p}\} \\ z_p^1 \leq z_{v_p}^1, \ z_{v_p}^1 \leq z_{\tilde{v}} & \text{if } \bar{w}_p = \{v_p\}, \ \forall p \in I(e) \setminus \{\tilde{p}\} \end{cases} \\
&\sum_{v \in U(e)} (1 - \delta_e(v)) z_v^1 + \sum_{p \in I(e) \setminus \{\tilde{p}\}} z_p^1 + z_{\tilde{p}} - z_e \leq (\omega(e) - 1) z_{\tilde{v}}.
\end{aligned} \tag{50}$$

We make use of the following claim to complete the proof of this theorem; we state this result without a proof as the proof as is similar to the proof of Claim 6.

Claim 8. *By projecting out z^1_p for all $p \in I(e) \cup U(e)$ from system* (50)*, we obtain*

$$\begin{aligned} &\begin{cases} z_p \leq 1 & \text{if } \bar{w}_p = \emptyset, \ \forall p \in I(e) \\ z_p \leq z_{v_p} & \text{if } \bar{w}_p = \{v_p\}, \ \forall p \in I(e) \end{cases} \\ &z_e \leq z_p \qquad \forall p \in I(e) \\ &\sum_{v \in U(e)} (1 - \delta_e(v)) z_v + \sum_{p \in I(e)} z_p - z_e \leq \omega(e) - 1. \end{aligned} \tag{51}$$

Characterization of MP$_G$. We now employ the results of Claims 7 and 8 to characterize MP$_G$ in the original space. Denote by $\tilde{E}(G)$ the set containing the sequence of nested edges of G containing $\tilde{v}$. The proof is by induction on the cardinality of $\tilde{E}(G)$. In the base case, we have $\tilde{E}(G) = \{e_0\}$. By definition of $\tilde{v}$, this implies that $E(G) = \{e_0\}$. Consider the system of inequalities defined by (50). By letting $e = e_0$, $\tilde{p} = \tilde{v}$, and $I(e_0) = V(G)$ which implies $\bar{w}_p = \emptyset$ for all $p \in I(e_0)$, these inequalities coincide with systems (25) and (26). Therefore, by Claim 8, in this case MP$_G$ is given by system (51), which coincides with system (19) with $I(e_0) = V$.

Now, suppose that $|\tilde{E}(G)| \geq 2$ and define $\{\tilde{e}\} := I(e_0) \cap \tilde{E}(G)$. Consider a running intersection ordering $\mathcal{O}(e_0)$ of the edges in $I(e_0)$ in which $\tilde{e}$ is the first element. The existence of such an ordering follows from Lemmata 1 and 4. Denote by w_e the intersection of each edge with all previous ones in $\mathcal{O}(e_0)$. Let $\bar{e}$ be the last element in $\mathcal{O}(e_0)$ and denote by $G_{\bar{e}}$ the section hypergraph of G induced by $\bar{e}$. Clearly, $\bar{e} \notin \tilde{E}(G)$ and $\bar{e}$ is an extremal-edge of H_{e_0}. Hence, by Claim 7, by projecting out z^1_v for all $v \in V(G_{\bar{e}}) \setminus w_{\bar{e}}$ and z^1_e for all $e \in E(G_{\bar{e}}) \setminus \{\bar{e}\}$ from inequalities of systems (25) and (26) containing these variables, we obtain system (47) together with inequalities (48) if $w_{\bar{e}} = \emptyset$, and inequalities (49) if $w_{\bar{e}} = \{v_{\bar{e}}\}$. Similarly, apply this projection recursively for each element $\hat{e}$ in $\mathcal{O}(e_0) \setminus \{\tilde{e}\}$ in a reverse order to project out z^1_v for all $v \in V(G_{\hat{e}}) \setminus w_{\hat{e}}$ and z^1_e for all $e \in E(G_{\hat{e}}) \setminus \{\hat{e}\}$, where $G_{\hat{e}}$ denotes the section hypergraph of G induced by $\hat{e}$.

Let G' denote the section hypergraph of G induced by $\tilde{e}$. Clearly, G' is an almost laminar hypergraph with $|\tilde{E}(G')| = |\tilde{E}(G)| - 1$. In addition, $w_{\tilde{e}} = \emptyset$ as by construction, $\tilde{e}$ is first element of $\mathcal{O}(e_0)$. Hence, by the induction hypothesis, projecting out z^1_p for all $p \in V(G') \cup E(G')$ gives system (19) with G replaced by G'. It can now be seen that the remaining inequalities containing variables z^1_p, $p \in I(e_0) \cup U(e_0) \setminus \{\tilde{e}\}$ coincide with system (50) by letting $e = e_0$ and $\tilde{p} = \tilde{e}$. Consequently, by projecting out these variables using Claim 8, we conclude that MP$_G$ is given by (19).

4.2 Proof of Theorem 2

In this proof we often consider β-cycles. It can be checked that a sequence $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_{t+1} = v_1$ is a β-cycle in G if and only if $t \geq 3$ and the edge e_i contains v_i, v_{i+1} and no other v_j, for $i = 1, \ldots, t$.

If $\bar{p} = \emptyset$, the result is obvious; thus, we assume that $\bar{p}$ is nonempty. Similarly, we assume that the sets $V(G) \setminus V(G_\omega)$ and $V(G) \setminus V(G_\alpha)$ are nonempty.

To proceed with the proof, we need a structural result regarding the hypergraph $\tilde{G}_\alpha = (V_\alpha, \tilde{E}_\alpha)$ obtained from $G_\alpha = (V_\alpha, E_\alpha)$ by removing edge $\bar{p}$, all the edges that strictly contain $\bar{p}$, and all the edges strictly contained in $\bar{p}$. Since G_α is almost laminar, every edge in $\tilde{E}_\alpha$ contains at most one node of $\bar{p}$. Let $w_1, \ldots, w_k$ be the nodes in $\bar{p}$. For every $i \in \{1, \ldots, p\}$, let U_i contain node w_i and the nodes $w \in V_\alpha$ for which there exists a chain in $\tilde{G}_\alpha$ from w_i to w.

Claim 9. *The sets $U_1, \ldots, U_k$ are pairwise disjoint.*

Proof of claim. First we show that no node w_i belongs to a set U_j, for distinct indices i, j in $\{1, \ldots, k\}$. By contradiction, assume that there exists a chain P in $\tilde{G}_\alpha$ from w_i to w_j. Without loss of generality, choose i, j, and P such that the length of P is minimal. We now show that $C = P, \bar{p}, w_i$ is a β-cycle in G_α. Since every edge in $\tilde{E}_\alpha$ contains at most one node of $\bar{p}$, the chain P must have length at least two. By the minimality assumption $\bar{p}$ contains only the first (w_i) and last (w_j) nodes of P. Again by minimality, each edge of P contains only the preceding and succeeding node of P. Hence $C = P, \bar{p}, w_i$ is a β-cycle in G_α, which is a contradiction.

Consider now a node $w \in V_\alpha$ that is not in $\bar{p}$. We show that w cannot belong to $U_i \cap U_j$, for distinct indices i, j in $\{1, \ldots, k\}$. By contradiction, assume that $w \in U_i \cap U_j$. Then there exists a chain P^i in $\tilde{G}_\alpha$ from w to w_i and a chain P^j in $\tilde{G}_\alpha$ from w_j to w. Without loss of generality, choose w, i, j, P^i, and P^j such that the sum of the lengths of P^i and P^j is minimal. We now show that $C = P^i, \bar{p}, P^j$ is a β-cycle in G_α. Note that all nodes of P^i (resp. P^j) except for w_i (resp. w_j) are not in $\bar{p}$, as otherwise such node $w_l \in \bar{p}$ would be in $U_l \cap U_i$ (resp. $U_l \cap U_j$). By the minimality assumption, each edge of P^i contains only the preceding and succeeding node of P^i. Symmetrically, each edge of P^j contains only the preceding and succeeding node of P^j. Again by minimality, no edge of P^i (resp. P^j) contains nodes of P^j (resp. P^i) different from w. Hence $C = P^i, \bar{p}, P^j$ is a β-cycle in G_α, which is a contradiction. ◇

To simplify the notation in the remainder of the proof, it will be useful to consider the nodes in $V_\alpha \setminus (\cup_{i=1}^k U_i)$ together with one of the sets $U_1, \ldots, U_k$, instead than on their own. For this reason we define the sets $W_i := U_i$, for $i = 1, \ldots, k-1$, and $W_k := W_k \cup (V_\alpha \setminus (\cup_{i=1}^k U_i))$.

Claim 10. *The sets $W_1, \ldots, W_k$ form a partition of V_α. Moreover, every edge of $\tilde{G}_\alpha$ is contained in exactly one of these sets.*

Proof of claim. Claim 9 directly implies that the sets $W_1, \ldots, W_k$ form a partition of V_α. By definition of the sets $U_1, \ldots, U_k$, every edge of $\tilde{G}_\alpha$ is either contained in one of these set, or it is contained in $V_\alpha \setminus (\cup_{i=1}^k U_i)$. Hence, every edge of $\tilde{G}_\alpha$ is contained in exactly one of the sets $W_1, \ldots, W_k$. ◇

In the next two claims we utilize Claim 10 to obtain vectors in $\mathcal{S}_G$ by combining a number of vectors in $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$. We now explain how we write a vector z in the space defined by G in the rest of the proof by partitioning its components in a number of subvectors. The vector $z_\cap$ contains the components of z corresponding to nodes and edges that are both in G_α and in G_ω (i.e., the nodes $w_1, \ldots, w_k$, the edge $\bar{p}$ and any other edge contained in $\bar{p}$). The vector z_0 contains the components of z corresponding to edges that are in G_α and strictly contain edge $\bar{p}$. For $i = 1, \ldots, k$, the vector z_i contains the components of z corresponding to nodes in $W_i \setminus \{w_i\}$ and edges contained in W_i. Finally, the vector z_{k+1} contains the components of z corresponding to nodes and edges in G_ω but not in G_α. Using these definitions, we can now write, up to reordering variables, $z = (z_0, z_1, \ldots, z_k, z_\cap, z_{k+1})$. Similarly, we can write a vector z in the space defined by G_α as $z = (z_0, z_1, \ldots, z_k, z_\cap)$, and a vector z in the space defined by G_ω as $z = (z_\cap, z_{k+1})$.

Claim 11. *Let $z^\alpha = (z_0^\alpha, z_1^\alpha, \ldots, z_k^\alpha, z_\cap^\alpha)$ be a vector in $\mathcal{S}_{G_\alpha}$, and let $z^\omega = (z_\cap^\omega, z_{k+1}^\omega)$ be a vector in $\mathcal{S}_{G_\omega}$ such that $z_{\bar{p}}^\alpha = z_{\bar{p}}^\omega = 1$. Then the vector $\tilde{z} = (z_0^\alpha, z_1^\alpha, \ldots, z_k^\alpha, z_\cap^\omega, z_{k+1}^\omega)$ is in $\mathcal{S}_G$.*

Proof of claim. To prove the claim, we show that for each edge e of G, we have $\tilde{z}_e = \prod_{v \in e} \tilde{z}_v$. First, we consider the edges of G_ω. For each edge e of G_ω, we have $\tilde{z}_e = z_e^\omega = \prod_{v \in e} z_v^\omega = \prod_{v \in e} \tilde{z}_v$. Next, we consider the edges of G_α. For each edge e of G_α, we have $\tilde{z}_e = z_e^\alpha = \prod_{v \in e} z_v^\alpha = \prod_{v \in e \setminus \bar{p}} z_v^\alpha \cdot \prod_{v \in e \cap \bar{p}} z_v^\alpha$. Note that, for every node $v \in \bar{p}$, we have $z_v^\alpha = z_v^\omega = 1$ since $z_{\bar{p}}^\alpha = z_{\bar{p}}^\omega = 1$. Hence we have $\tilde{z}_e = \prod_{v \in e \setminus \bar{p}} z_v^\alpha \cdot \prod_{v \in e \cap \bar{p}} z_v^\omega = \prod_{v \in e} \tilde{z}_v$. ◇

Claim 12. *Let* $z^{\alpha_1} = (z_0^{\alpha_1}, z_1^{\alpha_1}, \ldots, z_k^{\alpha_1}, z_\cap^{\alpha_1}), \ldots, z^{\alpha_k} = (z_0^{\alpha_k}, z_1^{\alpha_k}, \ldots, z_k^{\alpha_k}, z_\cap^{\alpha_k})$ *be* k *vectors in* $\mathcal{S}_{G_\alpha}$*, and let* $z^\omega = (z_\cap^\omega, z_{k+1}^\omega)$ *be a vector in* $\mathcal{S}_{G_\omega}$ *such that (1)* $z_{\bar{p}}^{\alpha_1} = \cdots = z_{\bar{p}}^{\alpha_k} = z_{\bar{p}}^\omega = 0$*, (2)* $z_{w_i}^{\alpha_i} = z_{w_i}^\omega$ *for every* $i = 1, \ldots, k$*. Then the vector* $\tilde{z} = (z_0^{\alpha_1}, z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_k^{\alpha_k}, z_\cap^\omega, z_{k+1}^\omega)$ *is in* $\mathcal{S}_G$*.*

Proof of claim. To prove the claim, we show that for each edge e of G, we have $\tilde{z}_e = \prod_{v\in e} \tilde{z}_v$. First, we consider the edges of G_ω. For each edge e in G_ω, we have $\tilde{z}_e = z_e^\omega = \prod_{v\in e} z_v^\omega = \prod_{v\in e} \tilde{z}_v$. Next, we consider the edges of G_α. Note that we have $z_0^{\alpha_1} = \cdots = z_0^{\alpha_k} = z_0^\omega = 0$ since $z_{\bar{p}}^{\alpha_1} = \cdots = z_{\bar{p}}^{\alpha_k} = z_{\bar{p}}^\omega = 0$. For each edge e contained in $\bar{p}$, we have $\tilde{z}_e = z_e^\omega = \prod_{v\in e} z_v^\omega = \prod_{v\in e} \tilde{z}_v$. For each edge e that strictly contains $\bar{p}$, we have $\tilde{z}_e = z_e^{\alpha_1} = 0$ since $z_0^{\alpha_1} = 0$; moreover $\prod_{v\in e} \tilde{z}_v \le \prod_{v\in\bar{p}} \tilde{z}_v = \prod_{v\in\bar{p}} z_v^\omega = z_{\bar{p}}^\omega = 0$ since $z_0^\omega = 0$. Finally, let e be an edge that contains at most one node of $\bar{p}$. We have that, by Claim 10, $e \subseteq W_i$, for some $i \in \{1, \ldots, k\}$ thus we have $\tilde{z}_e = z_e^{\alpha_i} = \prod_{v\in e} z_v^{\alpha_i}$. If $w_i \notin e$, then $z_v^{\alpha_i} = \tilde{z}_v$ for every $v \in e$, thus $\tilde{z}_e = \prod_{v\in e} \tilde{z}_v$. Otherwise, if $w_i \in e$, we have that $z_{w_i}^{\alpha_i} = z_{w_i}^\omega$, hence $\tilde{z}_e = z_{w_i}^\omega \cdot \prod_{v\in e\setminus\{w_i\}} z_v^{\alpha_i} = \prod_{v\in e} \tilde{z}_v$. ◇

We now proceed with the proof of the statement of the theorem. The inclusion $\operatorname{conv} \mathcal{S}_G \subseteq \operatorname{conv} \bar{\mathcal{S}}_{G_\alpha} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_\omega}$ clearly holds, since $\mathcal{S}_G \subseteq \bar{\mathcal{S}}_{G_\alpha} \cap \bar{\mathcal{S}}_{G_\omega}$. Thus, it suffices to show the reverse inclusion. Let $\hat{z} \in \operatorname{conv} \bar{\mathcal{S}}_{G_\alpha} \cap \operatorname{conv} \bar{\mathcal{S}}_{G_\omega}$. We will show that $\hat{z} \in \operatorname{conv} \mathcal{S}_G$.

By assumption, the vector $(\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_k, \hat{z}_\cap)$ is in $\operatorname{conv} \mathcal{S}_{G_\alpha}$. Thus, it can be written as a convex combination of points in $\mathcal{S}_{G_\alpha}$; i.e., there exists $\mu \ge 0$ with $\sum_{\alpha\in A} \mu_\alpha = 1$ such that

$$(\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_k, \hat{z}_\cap) = \sum_{\alpha\in A} \mu_\alpha (z_0^\alpha, z_1^\alpha, \ldots, z_k^\alpha, z_\cap^\alpha), \tag{52}$$

where the vectors $(z_0^\alpha, z_1^\alpha, \ldots, z_k^\alpha, z_\cap^\alpha)$, for $\alpha \in A$, belong to $\mathcal{S}_{G_\alpha}$. For each $i = 1, \ldots, k$, we partition the index set A into $A^{i,0} \cup A^{i,1}$, where $\alpha \in A^{i,1}$ if and only if $z_{w_i}^\alpha = 1$. Similarly, the vector $(\hat{z}_\cap, \hat{z}_{k+1})$ is in $\operatorname{conv} \mathcal{S}_{G_\omega}$ and it can be written as a convex combination of points in $\mathcal{S}_{G_\omega}$; i.e., there exists $\nu \ge 0$ with $\sum_{\omega\in\Omega} \nu_\omega = 1$ such that

$$(\hat{z}_\cap, \hat{z}_{k+1}) = \sum_{\omega\in\Omega} \nu_\omega (z_\cap^\omega, z_{k+1}^\omega), \tag{53}$$

where the vectors $(z_\cap^\omega, z_{k+1}^\omega)$, for $\omega \in \Omega$, belong to $\mathcal{S}_{G_\omega}$. We partition the index set Ω differently to how we partition A. Namely, we partition Ω into Ω^T, for $T \subseteq \bar{p}$, where $\omega \in \Omega^T$ if and only if for every $v \in \bar{p}$ we have $z_v^\omega = 1$ if and only if $v \in T$.

We now obtain some relations between the multipliers μ, ν, and the vector $\hat{z}$ that will be used in the remainder of the proof. By considering the component of (52) and of (53) corresponding to $\bar{p}$ we obtain

$$\begin{aligned} \hat{z}_{\bar{p}} &= \sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}} \mu_\alpha = \sum_{\omega\in\Omega^{\bar{p}}} \nu_\omega, \qquad \text{thus} \\ 1 - \hat{z}_{\bar{p}} &= \sum_{\alpha\in A\setminus(A^{1,1}\cap\cdots\cap A^{k,1})} \mu_\alpha = \sum_{T\subset\bar{p},\ \omega\in\Omega^T} \nu_\omega. \end{aligned} \tag{54}$$

By considering the component of (52) and of (53) corresponding to w_i, for $i = 1, \ldots, k$, we obtain

$$\begin{aligned} \hat{z}_{w_i} &= \sum_{\alpha\in A^{i,1}} \mu_\alpha = \sum_{T\subseteq\bar{p}\,:\, w_i\in T,\ \omega\in\Omega^T} \nu_\omega, \qquad \text{thus} \\ 1 - \hat{z}_{w_i} &= \sum_{\alpha\in A^{i,0}} \mu_\alpha = \sum_{T\subseteq\bar{p}\,:\, w_i\notin T,\ \omega\in\Omega^T} \nu_\omega. \end{aligned}$$

By defining, for $T \subset \bar{p}$,

$$\rho_T(w_i) := \begin{cases} \hat{z}_{w_i} - \hat{z}_{\bar{p}} & \text{if } w_i \in T, \\ 1 - \hat{z}_{w_i} & \text{if } w_i \notin T, \end{cases} \qquad \rho(T) := \prod_{i=1}^{k} \rho_T(w_i), \tag{55}$$

we obtain the following relation regarding multipliers μ:

$$\sum_{\alpha \in A^{i,\chi_T(w_i)} \setminus (A^{1,1} \cap \cdots \cap A^{k,1})} \mu_\alpha = \rho_T(w_i). \tag{56}$$

For multipliers ν we derive

$$\begin{aligned} \sum_{T \subset \bar{p}\,:\, w_i \in T,\ \omega \in \Omega^T} \nu_\omega &= \sum_{T \subseteq \bar{p}\,:\, w_i \in T,\ \omega \in \Omega^T} \nu_\omega - \sum_{\omega \in \Omega^{\bar{p}}} \nu_\omega = \hat{z}_{w_i} - \hat{z}_{\bar{p}}, \\ \sum_{T \subset \bar{p}\,:\, w_i \notin T,\ \omega \in \Omega^T} \nu_\omega &= \sum_{T \subseteq \bar{p}\,:\, w_i \notin T,\ \omega \in \Omega^T} \nu_\omega = 1 - \hat{z}_{w_i}. \end{aligned} \tag{57}$$

For every $\alpha \in A^{1,1} \cap \cdots \cap A^{k,1}$ and $\omega \in \Omega^{\bar{p}}$, we denote by $z^{\alpha,\omega} := (z_0^\alpha, z_1^\alpha, \ldots, z_k^\alpha, z_\cap^\omega, z_{k+1}^\omega)$, which is in $\mathcal{S}_G$ by Claim 11. For every $T \subset \bar{p}$, $\alpha_i \in A^{i,\chi_T(w_i)} \setminus (A^{1,1} \cap \cdots \cap A^{k,1})$, for $i = 1, \ldots, k$, and $\omega \in \Omega^T$, we denote by $z^{\alpha_1,\ldots,\alpha_k,\omega} := (z_0^{\alpha_1}, z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_k^{\alpha_k}, z_\cap^\omega, z_{k+1}^\omega)$. Note that the vector $z^{\alpha_1,\ldots,\alpha_k,\omega}$ is in $\mathcal{S}_G$ by Claim 12.

Claim 13. *The vector $\hat{z}$ can be written as $\hat{z}_{\bar{p}}\hat{z}^1 + (1 - \hat{z}_{\bar{p}})\hat{z}^0$, where $\hat{z}^1$ and $\hat{z}^0$ are defined as the following convex combination of vectors in $\mathcal{S}_G$:*

$$\hat{z}^1 := \sum_{\substack{\omega \in \Omega^{\bar{p}}, \\ \alpha \in A^{1,1} \cap \cdots \cap A^{k,1}}} \frac{\mu_\alpha \nu_\omega}{(\hat{z}_{\bar{p}})^2} \cdot z^{\alpha,\omega} \tag{58}$$

$$\hat{z}^0 := \sum_{\substack{T \subset \bar{p},\ \omega \in \Omega^T, \\ \alpha_i \in A^{i,\chi_T(w_i)} \setminus (A^{1,1} \cap \cdots \cap A^{k,1}),\, i=1,\ldots,k}} \frac{\mu_{\alpha_1} \cdots \mu_{\alpha_k} \nu_\omega}{(1 - \hat{z}_{\bar{p}})\rho(T)} \cdot z^{\alpha_1,\ldots,\alpha_k,\omega} \tag{59}$$

Proof of claim. Note that all the multipliers are nonnegative. We verify that they sum up to one. First consider the multipliers in (58). We obtain

$$\sum_{\substack{\omega \in \Omega^{\bar{p}}, \\ \alpha \in A^{1,1} \cap \cdots \cap A^{k,1}}} \frac{\mu_\alpha \nu_\omega}{(\hat{z}_{\bar{p}})^2} = \frac{1}{(\hat{z}_{\bar{p}})^2} \cdot \sum_{\omega \in \Omega^{\bar{p}}} \nu_\omega \cdot \sum_{\alpha \in A^{1,1} \cap \cdots \cap A^{k,1}} \mu_\alpha = 1,$$

where the last equation follows from (54). Next consider the multipliers in (59). We have

$$\begin{aligned} &\sum_{\substack{T \subset \bar{p},\ \omega \in \Omega^T, \\ \alpha_i \in A^{i,\chi_T(w_i)} \setminus (A^{1,1} \cap \cdots \cap A^{k,1}),\, i=1,\ldots,k}} \frac{\mu_{\alpha_1} \cdots \mu_{\alpha_k} \nu_\omega}{(1 - \hat{z}_{\bar{p}})\rho(T)} = \\ &= \frac{1}{1 - \hat{z}_{\bar{p}}} \cdot \sum_{T \subset \bar{p},\ \omega \in \Omega^T} \frac{\nu_\omega}{\rho(T)} \cdot \prod_{i=1}^{k} \left(\sum_{\alpha_i \in A^{i,\chi_T(w_i)} \setminus (A^{1,1} \cap \cdots \cap A^{k,1})} \mu_{\alpha_i} \right) \\ &= \frac{1}{1 - \hat{z}_{\bar{p}}} \cdot \sum_{T \subset \bar{p},\ \omega \in \Omega^T} \nu_\omega = 1, \end{aligned}$$

where the second equation holds by (55) and (56), and the last equation follows from (54).

In the remainder of the proof of we show that $\hat{z}_{\bar{p}}\hat{z}^1 + (1-\hat{z}_{\bar{p}})\hat{z}^0 = \hat{z}$. First, we consider components $\bullet \in \{\cap, k+1\}$. We calculate $\hat{z}_{\bar{p}}\hat{z}^1_\bullet$ using (58).

$$\hat{z}_{\bar{p}}\hat{z}^1_\bullet = \frac{1}{\hat{z}_{\bar{p}}} \cdot \sum_{\substack{\omega\in\Omega^{\bar{p}},\\ \alpha\in A^{1,1}\cap\cdots\cap A^{k,1}}} \mu_\alpha \nu_\omega z^\omega_\bullet = \frac{1}{\hat{z}_{\bar{p}}} \cdot \sum_{\omega\in\Omega^{\bar{p}}} \nu_\omega z^\omega_\bullet \cdot \sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}} \mu_\alpha = \sum_{\omega\in\Omega^{\bar{p}}} \nu_\omega z^\omega_\bullet,$$

where the last equation holds by (54). Next, we calculate $(1-\hat{z}_{\bar{p}})\hat{z}^0_\bullet$ using (59).

$$\begin{aligned}(1-\hat{z}_{\bar{p}})\hat{z}^0_\bullet &= \sum_{\substack{T\subset\bar{p},\ \omega\in\Omega^T,\\ \alpha_i\in A^{i,\chi_T(w_i)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1}),\, i=1,\ldots,k}} \frac{\mu_{\alpha_1}\cdots\mu_{\alpha_k}\nu_\omega}{\rho(T)} \cdot z^\omega_\bullet \\ &= \sum_{T\subset\bar{p},\ \omega\in\Omega^T} \frac{\nu_\omega}{\rho(T)} \cdot z^\omega_\bullet \cdot \prod_{i=1}^k \Bigg(\sum_{\alpha_i\in A^{i,\chi_T(w_i)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})} \mu_{\alpha_i} \Bigg) \\ &= \sum_{T\subset\bar{p},\ \omega\in\Omega^T} \nu_\omega z^\omega_\bullet,\end{aligned}$$

where in the third equation we used (55) and (56). We obtain that

$$\hat{z}_{\bar{p}}\hat{z}^1_\bullet + (1-\hat{z}_{\bar{p}})\hat{z}^0_\bullet = \sum_{\omega\in\Omega^{\bar{p}}} \nu_\omega z^\omega_\bullet + \sum_{T\subset\bar{p},\ \omega\in\Omega^T} \nu_\omega z^\omega_\bullet = \sum_{\omega\in\Omega} \nu_\omega z^\omega_\bullet = \hat{z}_\bullet,$$

where in the last equation we used (53).

To simplify our calculation of $\hat{z}_{\bar{p}}\hat{z}^1 + (1-\hat{z}_{\bar{p}})\hat{z}^0$ for the remaining components $\bullet \in \{0, 1, \ldots, k\}$, we calculate $\hat{z}_{\bar{p}}\hat{z}^1$ using (58). We obtain

$$\hat{z}_{\bar{p}}\hat{z}^1_\bullet = \frac{1}{\hat{z}_{\bar{p}}} \cdot \sum_{\substack{\omega\in\Omega^{\bar{p}},\\ \alpha\in A^{1,1}\cap\cdots\cap A^{k,1}}} \mu_\alpha \nu_\omega z^\alpha_\bullet = \frac{1}{\hat{z}_{\bar{p}}} \cdot \sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}} \mu_\alpha z^\alpha_\bullet \cdot \sum_{\omega\in\Omega^{\bar{p}}} \nu_\omega = \sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}} \mu_\alpha z^\alpha_\bullet, \quad (60)$$

where the last equation holds by (54).

We now consider the components z_0 and we show that $\hat{z}_{\bar{p}}\hat{z}^1_0 + (1-\hat{z}_{\bar{p}})\hat{z}^0_0 = \hat{z}_0$. We will be using the fact that for each $\alpha \in A \setminus (A^{1,1}\cap\cdots\cap A^{k,1})$, we have that $z^\alpha_0 = 0$ since each component corresponds to an edge that strictly contains edge $\bar{p}$ and at least one node in $\bar{p}$ has its component in z^α_0 equal to zero. First we show that $\hat{z}^0_0 = 0$. For each vector $z_0^{\alpha_1,\ldots,\alpha_k,\omega}$ in the sum (59), we have $z_0^{\alpha_1,\ldots,\alpha_k,\omega} = z_0^{\alpha_1}$ and $\alpha_1 \in A \setminus (A^{1,1}\cap\cdots\cap A^{k,1})$, thus $z_0^{\alpha_1,\ldots,\alpha_k,\omega} = 0$ and $\hat{z}^0_0 = 0$. We obtain

$$\hat{z}_{\bar{p}}\hat{z}^1_0 + (1-\hat{z}_{\bar{p}})\hat{z}^0_0 = \hat{z}_{\bar{p}}\hat{z}^1_0 = \sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}} \mu_\alpha z^\alpha_0 = \sum_{\alpha\in A} \mu_\alpha z^\alpha_0 = \hat{z}_0,$$

where the second equation holds by (60), and the third equation follows by the observation above.

Finally, we consider the components z_j, for $j = 1, \ldots, k$, and we show that $\hat{z}_{\bar{p}}\hat{z}^1_j + (1-\hat{z}_{\bar{p}})\hat{z}^0_j = \hat{z}_j$.

We calculate $(1-\hat{z}_{\bar{p}})\hat{z}_j^0$ using (59).

$$
\begin{aligned}
(1-\hat{z}_{\bar{p}})\hat{z}_j^0 &= \sum_{\substack{T\subset\bar{p},\ \omega\in\Omega^T,\\ \alpha_i\in A^{i,\chi_T(w_i)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1}),\, i=1,\ldots,k}} \frac{\mu_{\alpha_1}\cdots\mu_{\alpha_k}\nu_\omega}{\rho(T)}\cdot z_j^{\alpha_j}\\
&= \sum_{\substack{T\subset\bar{p},\ \omega\in\Omega^T,\\ \alpha_j\in A^{j,\chi_T(w_j)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}} \frac{\mu_{\alpha_j}\nu_\omega}{\rho(T)}\cdot z_j^{\alpha_j}\cdot \prod_{i\in\{1,\ldots,k\}\setminus\{j\}}\Bigg(\sum_{\alpha_i\in A^{i,\chi_T(w_i)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}\mu_{\alpha_i}\Bigg)\\
&= \sum_{\substack{T\subset\bar{p},\ \omega\in\Omega^T,\\ \alpha_j\in A^{j,\chi_T(w_j)}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}} \frac{\mu_{\alpha_j}\nu_\omega}{\rho_T(w_j)}\cdot z_j^{\alpha_j}\\
&= \sum_{\substack{T\subset\bar{p}:\,\omega_j\in T,\ \omega\in\Omega^T,\\ \alpha_j\in A^{j,1}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}} \frac{\mu_{\alpha_j}\nu_\omega}{\hat{z}_{w_j}-\hat{z}_{\bar{p}}}\cdot z_j^{\alpha_j} + \sum_{\substack{T\subset\bar{p}:\,\omega_j\notin T,\ \omega\in\Omega^T,\\ \alpha_j\in A^{j,0}}} \frac{\mu_{\alpha_j}\nu_\omega}{1-\hat{z}_{w_j}}\cdot z_j^{\alpha_j}\\
&= \frac{1}{\hat{z}_{w_j}-\hat{z}_{\bar{p}}}\cdot \sum_{\alpha_j\in A^{j,1}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}\mu_{\alpha_j}z_j^{\alpha_j}\cdot \sum_{T\subset\bar{p}:\,\omega_j\in T,\ \omega\in\Omega^T}\nu_\omega +\\
&\quad + \frac{1}{1-\hat{z}_{w_j}}\cdot\sum_{\alpha_j\in A^{j,0}}\mu_{\alpha_j}z_j^{\alpha_j}\cdot\sum_{T\subset\bar{p}:\,\omega_j\notin T,\ \omega\in\Omega^T}\nu_\omega\\
&= \sum_{\alpha_j\in A^{j,1}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}\mu_{\alpha_j}z_j^{\alpha_j} + \sum_{\alpha_j\in A^{j,0}}\mu_{\alpha_j}z_j^{\alpha_j},
\end{aligned}
$$

where in the third equation we used (55) and (56), in the fourth equation we used the definition of $\rho_T(w_j)$ in (55), and in the sixth equation we used (57). Using the obtained expression and (60), we have that $\hat{z}_{\bar{p}}\hat{z}_j^1+(1-\hat{z}_{\bar{p}})\hat{z}_j^0$ equals

$$
\sum_{\alpha\in A^{1,1}\cap\cdots\cap A^{k,1}}\mu_\alpha z_j^\alpha + \sum_{\alpha_j\in A^{j,1}\setminus(A^{1,1}\cap\cdots\cap A^{k,1})}\mu_{\alpha_j}z_j^{\alpha_j} + \sum_{\alpha_j\in A^{j,0}}\mu_{\alpha_j}z_j^{\alpha_j} = \sum_{\alpha\in A}\mu_\alpha z_j^\alpha = \hat{z}_j,
$$

where the last equation follows by (52). ◇

4.3 Proof of Theorem 3

Consider a β-acyclic hypergraph $G=(V,E)$ with the simple intersection property. By Lemma 4 there exists a running intersection ordering $\mathcal{O}=\bar{e}_1,\ldots,\bar{e}_\kappa$ of the set of maximal edges of G. Let $G_{\bar{e}_\kappa}$ denote the subhypergraph of G induced by $\bar{e}_\kappa$. Since G is a β-acyclic hypergraph with the simple intersection property, by Observation 1, $G_{\bar{e}_\kappa}$ is an almost laminar hypergraph. Now consider the hypergraph $G^+=(V,E^+)$, where E^+ is defined by (20). We define G^1_α as the section hypergraph of G^+ induced by $\bar{e}_\kappa$, and G^1_ω as the section hypergraph of G^+ induced by $\cup_{E^+\setminus E(G^1_\alpha)}e$. It is simple to check that G^1_α is a partial hypergraph of the almost laminar hypergraph $G_{\bar{e}_\kappa}$. Hence, G^1_α is an almost laminar hypergraph as well. In addition, both G^1_α and G^1_ω are different from G^+ and we have $G^1_\alpha\cup G^1_\omega=G^+$, $G^1_\alpha\cap G^1_\omega=N(\bar{e}_\kappa)$, where the set $N(\bar{e}_\kappa)$ is defined in (2). Finally, by construction, $N(\bar{e}_\kappa)\in E^+$. Thus all assumptions of Theorem 2 are satisfied and the set $\mathcal{S}_{G^+}$ is decomposable into $\mathcal{S}_{G^1_\alpha}$ and $\mathcal{S}_{G^1_\omega}$. As G^1_α is an almost laminar hypergraph, $\text{MP}_{G^1_\alpha}$ is given by Theorem 1.

Now define $G^+_{\setminus\kappa}:=G^1_\omega$ and consider the edge $\bar{e}_{\kappa-1}$, that is, the element of $\mathcal{O}$ before $\bar{e}_\kappa$. Let $G_{\bar{e}_{\kappa-1}}$ denote the subhypergraph of G induced by $\bar{e}_{\kappa-1}$. Again, by Observation 1, $G_{\bar{e}_{\kappa-1}}$ is an almost laminar hypergraph. Define G^2_α as the section hypergraph of $G^+_{\setminus\kappa}$ induced by $\bar{e}_{\kappa-1}$ and G^2_ω as the section hypergraph of $G^+_{\setminus\kappa}$ induced by $\cup_{E(G^+_{\setminus\kappa})\setminus E(G^2_\alpha)} e$. The hypergraph G^2_α is a partial hypergraph of $G_{\bar{e}_{\kappa-1}}$ and as a result is an almost laminar hypergraph as well. Similarly, we can verify that all assumptions are Theorem 2 are satisfied and the set $\mathcal{S}_{G^+\setminus\kappa}$ is decomposable into $\mathcal{S}_{G^2_\alpha}$ and $\mathcal{S}_{G^2_\omega}$. By a recursively application of this argument for all elements of $\mathcal{O}$ in the reverse order, we conclude that the multilinear set $\mathcal{S}_{G^+}$ is decomposable into the sets $\mathcal{S}_{G^i_\alpha}$, $i = 1, \ldots, \kappa$, where G^i_α is the section hypergraph of G^+ induced by $\bar{e}_{\kappa-i+1}$, which as detailed above is an almost laminar hypergraph with the corresponding multilinear polytope given by Theorem 1.

4.4 Proof of Theorem 4

Let $G = (V, E)$ be a β-acyclic hypergraph with the simple intersection property. The proof is by induction on the number of maximal edges of G. If G has one maximal edge, then the proof follows from Observation 1 and Corollary 3. Hence suppose that G has κ maximal edges for some $\kappa \geq 2$. By Lemma 4, there exists a running intersection ordering $\mathcal{O}$ of the set of maximal edges of G.

Lifting and decomposition. Denote by $\tilde{e}$ the last element of $\mathcal{O}$ and define $\bar{p} := N(\tilde{e})$. Let $G^+ = (V, E^+)$ be the hypergraph obtained from G by adding $\bar{p}$ to E if $\bar{p} \notin V \cup E$; that is, let $E^+ = E \cup \{\bar{p}\}$ if $\bar{p} \notin V \cup E$ and let $E^+ = E$, otherwise. Denote by G_α the section hypergraph of G^+ induced by $\tilde{e}$, and denote by G_ω the section hypergraph of G induced by $\cup_{E^+\setminus E(G_\alpha)} e$. As we detailed in the proof of Theorem 3, by Theorem 2, the multilinear set $\mathcal{S}_{G^+}$ is decomposable into multilinear sets $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$. As we argued in the proof of Theorem 3, G_α is an almost laminar hypergraph. Hence, by Corollary 3 we have $\text{MP}_{G_\alpha} = \text{MP}^{\text{RI}}_{G_\alpha}$.

Now consider the hypergraph G_ω. First note that G_ω has $\kappa - 1$ maximal edges that are different from $\tilde{e}$. We show that G_ω is a β-acyclic with the simple intersection property. If $\bar{p} \in V \cup E$, then G_ω is a partial hypergraph of G and hence the statement follows trivially. Hence, suppose that $\bar{p} \notin V \cup E$. It is simple to see that G_ω is the subhypergraph of G induced by $\cup_{e\in\bar{E}\setminus\{\tilde{e}\}} e$, where $\bar{E}$ denotes the set of maximal edges of G. Since G is β-acyclic, by Lemma 3, G_ω is β-acyclic as well. To show that G_ω has the simple intersection property, we need to show that exist no three edges $e_0, e_1, e_2 \in E(G_\omega)$ such that $|e_0 \cap e_1 \cap e_2| \geq 2$, $(e_0 \cap e_1) \setminus e_2 \neq \emptyset$, and $(e_0 \cap e_2) \setminus e_1 \neq \emptyset$. To obtain a contradiction, suppose that such three edges exist. Again, one of these edges, say e_0 must be the edge $\bar{p}$, since by assumption G has the simple intersection property. Since $\tilde{e} \cap \cup_{e\in E(G_\omega)} e = \bar{p}$, it follows that, the three edges $\tilde{e}$, e_1 and e_2 in G satisfy $|\tilde{e} \cap e_1 \cap e_2| \geq 2$, $(\tilde{e} \cap e_1) \setminus e_2 \neq \emptyset$, and $(\tilde{e} \cap e_2) \setminus e_1 \neq \emptyset$, which is in contradiction with the assumption that G has the simple intersection property. Hence, G_ω is a β-acyclic with the simple intersection property and by the induction hypothesis we have $\text{MP}_{G_\omega} = \text{MP}^{\text{RI}}_{G_\omega}$, which together with $\text{MP}_{G_\alpha} = \text{MP}^{\text{RI}}_{G_\alpha}$ and the decomposability of $\mathcal{S}_{G^+}$ into $\mathcal{S}_{G_\alpha}$ and $\mathcal{S}_{G_\omega}$, implies $\text{MP}_{G^+} = \text{MP}^{\text{RI}}_{G^+}$.

If $G = G^+$; that is if $\bar{p} \in V \cup E$, we obtain $\text{MP}_G = \text{MP}^{\text{RI}}_G$ and this completes the proof. Henceforth, assume that $\bar{p} \notin V \cup E$. To obtain MP_G, it suffices to project out the auxiliary variable $z_{\bar{p}}$ from the facet-description of MP_{G^+}. In the following, we perform this projection using Fourier-Motzkin elimination.

Projection. First consider an inequality in the description of $\text{MP}^{\text{RI}}_{G^+}$ that does not contain $z_{\bar{p}}$. Clearly, the support hypergraph of such an inequality is a partial hypergraph of G. The following claim establishes that this inequality is also present in the description MP^{RI}_G.

Claim 14. *Let G' be a partial hypergraph of G. Then all inequalities defining $MP^{RI}_{G'}$ are also present in the system defining MP^{RI}_G.*

Proof of claim. Clearly, $\mathrm{MP}^{\mathrm{LP}}_G$ contains all inequalities present in the description of $\mathrm{MP}^{\mathrm{LP}}_{G'}$, since the standard linearization of a multilinear set is obtained by intersecting the multilinear polytopes of each edge of the corresponding hypergraph and we have $E(G') \subset E(G)$. In addition, by definition of running intersection inequalities, every running intersection inequality for $\mathcal{S}_{G'}$ is also a running intersection inequality for $\mathcal{S}_G$, as again $E(G') \subset E(G)$. Hence, all inequalities defining $\mathrm{MP}^{\mathrm{RI}}_{G'}$ are also present in $\mathrm{MP}^{\mathrm{RI}}_G$. ⋄

To complete the proof, we need to show that by projecting out $z_{\bar{p}}$ from the remaining inequalities of $\mathrm{MP}^{\mathrm{RI}}_{G^+}$, we obtain valid inequalities for $\mathrm{MP}^{\mathrm{RI}}_G$. First, consider MP_{G_α}; denote by $\bar{e}$ the edge of G_α such that $\bar{p} \in I(\bar{e})$; the uniqueness of $\bar{e}$ follows from the fact that G_α is an almost laminar hypergraph. By Theorem 1, $z_{\bar{p}}$ appears in the following inequalities, which we will refer to as system (I) in the rest of the proof:

$$-z_p + z_{\bar{p}} \leq 0 \qquad \forall p \in I(\bar{p}) \tag{61}$$

$$-z_{\bar{p}} + z_{\bar{e}} \leq 0 \tag{62}$$

$$\sum_{v \in \bar{p}} (1 - \delta_{\bar{p}}(v)) z_v + \sum_{e \in I(\bar{p}) \cap E} z_e - z_{\bar{p}} \leq \omega(\bar{p}) - 1 \tag{63}$$

$$\sum_{v \in \bar{e}} (1 - \delta_{\bar{e}}(v)) z_v + \sum_{e \in I(\bar{e}) \cap E} z_e - z_{\bar{e}} \leq \omega(\bar{e}) - 1. \tag{64}$$

Now consider the polytope $\mathrm{MP}_{G_\omega} = \mathrm{MP}^{\mathrm{RI}}_{G_\omega}$. As we showed earlier, the hyperpgraph G_ω is β-acyclic with the simple intersection property. Hence, its running intersection inequalities are of the form (18). Let $\mathcal{E}_{\bar{p}}$ be the set containing all subsets of edges $E_{\bar{p}}$ in G_ω such that the center edge $\bar{p}$ together with neighbors e, $\forall e \in E_{\bar{p}}$ satisfy Conditions (i) and (ii) of Proposition 3. Note that $\mathcal{E}_{\bar{p}}$ contains the empty set. Let $\hat{E}$ denote the set of all edges $\hat{e}$ of G_ω such that $|\bar{p} \cap \hat{e}| \geq 2$. For each $\hat{e} \in \hat{E}$, denote by $\mathcal{E}_{\hat{e}}$ the set containing all subsets of edges $E_{\hat{e}}$ in G_ω such that $\bar{p} \in E_{\hat{e}}$ and the center edge $\hat{e}$ with neighbors e, $\forall e \in E_{\hat{e}}$ satisfy Conditions (i) and (ii) of Proposition 3. Denote by $\omega(E_{\bar{p}})$ (resp. $\omega(E_{\hat{e}})$) the number of connected components in the hypergraph with the node set $\bar{p}$ (resp. $\hat{e}$) and the edge set $\{\bar{p} \cap e, \forall e \in E_{\bar{p}}\}$ (resp. $\{\hat{e} \cap e, \forall e \in E_{\hat{e}}\}$). Finally, for each $v \in \bar{p}$ (resp. $v \in \hat{e}$) denote by $\delta_{E_{\bar{p}}}(v)$ (resp. $\delta_{E_{\hat{e}}}(v)$) the number of edges in $E_{\bar{p}}$ (resp. $E_{\hat{e}}$) containing v. Then, the inequalities of $\mathrm{MP}^{\mathrm{RI}}_{G_\omega}$ containing $z_{\bar{p}}$ are given by:

$$-z_p + z_{\bar{p}} \leq 0 \qquad \forall p \in I(\bar{p}) \tag{65}$$

$$\sum_{v \in \bar{p}} (1 - \delta_{E_{\bar{p}}}(v)) z_v + \sum_{e \in E_{\bar{p}}} z_e - z_{\bar{p}} \leq \omega(E_{\bar{p}}) - 1 \qquad \forall E_{\bar{p}} \in \mathcal{E}_{\bar{p}} \tag{66}$$

$$\sum_{v \in \hat{e}} (1 - \delta_{E_{\hat{e}}}(v)) z_v + \sum_{e \in E_{\hat{e}}} z_e - z_{\hat{e}} \leq \omega(E_{\hat{e}}) - 1 \qquad \forall \hat{e} \in \hat{E}, \forall E_{\hat{e}} \in \mathcal{E}_{\hat{e}}. \tag{67}$$

In the remainder of the proof, we will refer to the inequalities (65)–(67) as system (II).

Now consider the system of linear inequalities (I)–(II). We eliminate $z_{\bar{p}}$ from this system using Fourier-Motzkin elimination. First suppose that we select two inequalities from system (I). Denote by G'_α the hypergraph obtained by removing the edge $\bar{p}$ from G_α. It then follows that the inequality $az \leq \alpha$ obtained as a result of such projection is valid for $\mathrm{MP}_{G'_\alpha}$. Since G'_α is an almost laminar hypergraph, by Corollary 3, we have $\mathrm{MP}_{G'_\alpha} = \mathrm{MP}^{\mathrm{RI}}_{G'_\alpha}$. Finally, since G'_α is a partial hypergraph of G, by Claim 14, $az \leq \alpha$ is a valid inequality for $\mathrm{MP}^{\mathrm{RI}}_G$. Similarly, we argue that by projecting out $z_{\bar{p}}$ from two inequalities of system (II), we obtain an inequality that is valid for $\mathrm{MP}^{\mathrm{RI}}_G$. To see this, observe that the hypergraph G'_ω obtained by removing $\bar{p}$ from G_ω is β-acyclic with the simple intersection property and has $\kappa - 1$ maximal edges for which by the induction hypothesis we have $\mathrm{MP}_{G'_\omega} = \mathrm{MP}^{\mathrm{RI}}_{G'_\omega}$. Therefore, it suffices to examine inequalities obtained by projecting out $z_{\bar{p}}$ starting from two inequalities one of which is only present in system (I) while the other one is only present in system (II).

We start by selecting one inequality in (61) from system (I). Clearly, this inequality is identical to inequality (65) present in system (II). Hence, by the above discussion, we do not need to consider inequalities (61). Next, consider inequality (62) from system (I). Since the coefficient of $z_{\bar{p}}$ in (62) is negative, it suffices to consider inequalities (65) and (67) from system (II). In addition, we do not need to consider (65) since it is already present system (I). By summing inequalities (62) and (67), for each $\hat{e} \in \hat{E}$ and each $E_{\hat{e}} \in \mathcal{E}_{\hat{e}}$ we obtain

$$\sum_{v \in \hat{e}} (1 - \delta_{E_{\hat{e}}}(v)) z_v + \sum_{e \in E_{\hat{e}} \setminus \{\bar{p}\}} z_e + z_{\bar{e}} - z_{\hat{e}} \le \omega(E_{\hat{e}}) - 1. \tag{68}$$

We claim that inequality (68) is a running intersection inequality of the form (18) centered at $\hat{e}$ with neighbors $E'_{\hat{e}} := (E_{\hat{e}} \setminus \{\bar{p}\}) \cup \{\bar{e}\}$. As before, let $\delta_{E'_{\hat{e}}}(v)$ denote the number of edges in $E'_{\hat{e}}$ containing the node $v \in \hat{e}$ and denote by $\omega(E'_{\hat{e}})$ the number of connected components in the hypergraph with the node set $\hat{e}$ and the edge set $\{\hat{e} \cap e, \forall e \in E'_{\hat{e}}\}$. For each $\hat{e} \in \hat{E}$ and each $E_{\hat{e}} \in \mathcal{E}_{\hat{e}}$, we have $\hat{e} \cap \bar{p} = \hat{e} \cap \bar{e}$ and $e \cap \bar{p} = e \cap \bar{e}$ for all $e \in E_{\hat{e}}$, as by definition $\bar{p} = N(\tilde{e})$, $\bar{e} \subseteq \tilde{e}$, $\bar{e} \supset \bar{p}$, $\hat{e} \not\subseteq \tilde{e}$, and $e \not\subseteq \tilde{e}$ for all $e \in E_{\hat{e}}$. This implies that conditions (i) and (ii) of Proposition 3 are satisfied for $\hat{e}$, $e \in E'_{\hat{e}}$, . Moreover, $\delta_{E_{\hat{e}}}(v) = \delta_{E'_{\hat{e}}}(v)$ for all $v \in \hat{e}$ and $\omega(E_{\hat{e}}) = \omega(E'_{\hat{e}})$. It then follows that for each $\hat{e} \in \hat{E}$ and each $E_{\hat{e}} \in \mathcal{E}_{\hat{e}}$, inequality (68) is a running intersection inequality of the form (18) is therefore present in $\mathrm{MP}^{\mathrm{RI}}_G$.

By construction, there exists a set $E_{\bar{p}} \in \mathcal{E}_{\bar{p}}$ such that $E_{\bar{p}} = I(\bar{p}) \cap E$. Therefore, inequalities (63) are implied by inequalities (66) and as a result we do not need to consider these inequalities. Hence we proceed with inequalities (64) from system (I). Since the coefficient of $z_{\bar{p}}$ in (64) is positive, it suffices to consider inequalities (66) from system (II). By summing inequalities (64) and (66), for each $E_{\bar{p}} \in \mathcal{E}_{\bar{p}}$ and defining $E_{\bar{e}} := E_{\bar{p}} \cup ((I(\bar{e}) \setminus \{\bar{p}\}) \cap E)$, we get:

$$\sum_{v \in \bar{e}} (1 - \delta_{\bar{e}}(v)) z_v + \sum_{v \in \bar{p}} (1 - \delta_{E_{\bar{p}}}(v)) z_v + \sum_{e \in E_{\bar{e}}} z_e - z_{\bar{e}} \le \omega(\bar{e}) + \omega(E_{\bar{p}}) - 2. \tag{69}$$

For each $v \in \bar{e}$, denote by $\delta_{E_{\bar{e}}}(v)$ the number of edges in $E_{\bar{e}}$ containing v and denote by $\omega(E_{\bar{e}})$ the number of connected components of the hypergraph $(\bar{e}, \tilde{E})$, where $\tilde{E} = \{e \cap \bar{e} : e \in E_{\bar{e}}\}$. It can be checked that $\omega(E_{\bar{e}}) = \omega(\bar{e}) + \omega(E_{\bar{p}}) - 1$. Clearly, for any node $v \in \bar{e} \setminus \bar{p}$ we have $\delta_{E_{\bar{e}}}(v) = \delta_{\bar{e}}(v)$. Now consider a node $v \in \bar{p}$; since $\bar{p} \in I(\bar{e})$ but $\bar{p} \notin E_{\bar{e}}$, we have $\delta_{E_{\bar{e}}}(v) = \delta_{E_{\bar{p}}}(v) + \delta_{\bar{e}}(v) - 1$. Since $\bar{e} \supset \bar{p}$, inequality (69) can be equivalently written as

$$\sum_{v \in \bar{e}} (1 - \delta_{E_{\bar{e}}}(v)) z_v + \sum_{e \in E_{\bar{e}}} z_e - z_{\bar{e}} \le \omega(E_{\bar{e}}) - 1. \tag{70}$$

To complete the proof, we need to show that $\bar{e}, e : e \in E_{\bar{e}}$ satisfy conditions (i) and (ii) of Proposition 3: condition (i) is clearly satisfied as $\bar{e} \supset e$ for all $e \in I(\bar{e}) \cap E$ and $|\bar{e} \cap e| \ge 2$ for all $e \in E_{\bar{p}}$ since $|\bar{p} \cap e| \ge 2$ for all $e \in E_{\bar{p}}$ and $\bar{e} \supset \bar{p}$. To demonstrate the validity of Condition (ii), we need to show that $e \cap \bar{e} \not\subseteq e' \cap \bar{e}$ for all $e, e' \in E_{\bar{e}}$. By definition $|e \cap e'| \le 1$ for all $e, e' \in I(\bar{e}) \cap E$; moreover, by construction $e \cap \bar{p} = e \cap \bar{e}$ for all $e \in E_{\bar{p}}$ and $e \cap \bar{p} \not\subseteq e' \cap \bar{p}$ for all $e, e' \in E_{\bar{p}}$. Finally $|e \cap e'| \le 1$ for all $e \in (I(\bar{e}) \setminus \{\bar{p}\}) \cap E$ and for all $e' \in E_{\bar{p}}$ as $|e \cap \bar{p}| \le 1$ for all $e \in (I(\bar{e}) \setminus \{\bar{p}\}) \cap E$ and by definition $\bar{p} = N(\tilde{e})$. Therefore, for each $E_{\bar{p}} \in \mathcal{E}_{\bar{p}}$, inequality (70) is a running intersection inequality of the form (18) centered at $\bar{e}$ with neighbors e, $e \in E_{\bar{e}}$ and hence is present in $\mathrm{MP}^{\mathrm{RI}}_G$.

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